

Liouville Super Gravity. Ramond sector

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2505.23122 [hep-th]

18 September 2025, Montpellier

Outline

- Introduction (the broader context of problems related to MLG)
- Continuous description of MLG
- Dual discrete formulation
- The observables: Amplitudes (sphere and one loop)
- Supersymmetric extension.
We analyze the **Ramond sector**.
The new progress:
 - 1) We clarified the **structure of the physical fields**.
 - 2) We showed that higher equations of motion imply a certain **relation between tachyonic fields and elements of the ground ring**. This relation allows to compute **integrals over moduli space that define gravitational amplitudes**.

Why 2D gravity?

- 1) It is simple enough, so it can serve as a laboratory for quantum gravity — exact computations are possible.
- 2) It is rich enough to connect many different subjects as string theory, matrix models, topological field theory, integrable hierarchies, etc.

Introduction

The Landscape. MLG is not the unique theory of gravity in low dimension (2D), there are other models of gravity in low dimensions which should be mentioned:

- Minimal Liouville Gravity. Its action consists of three parts:
 $S_{\text{MLG}} = S_{\text{Liouville}} + S_{\text{matter}} + S_{\text{ghosts}}$, where the Liouville term is

$$S_{\text{Liouville}} = \frac{1}{4\pi} \int d^2x \sqrt{g} ((\nabla \phi_L)^2 + QR\phi_L + 4\pi\mu e^{2b\phi_L})$$

- JT gravity (Dilaton ϕ_{JT} couples to curvature and gives dynamics to the boundary, described by Schwarzian action of the boundary reparametrizations $f(t)$)

$$S_{\text{JT}} = \frac{1}{2} \int d^2x \sqrt{g} \phi_{\text{JT}} (R + 2)$$

Here ϕ_{JT} is the dilaton field. It enforces constant negative curvature, $R = -2$, and gives rise to Schwarzian boundary dynamics. The metric itself has no local degrees of freedom, but the dilaton introduces boundary dynamics that turn out to be extremely rich. The metric near the boundary is fixed up to a reparametrization (a map from the boundary circle to itself). The dilaton controls the proper length of this boundary and makes those reparametrizations dynamical. After integrating out the bulk, what remains is an effective action for the boundary reparametrization $f(t)$. Note that ϕ_{JT} is conceptually different from ϕ_L :

- 1) ϕ_L : the Liouville conformal factor, a dynamical scalar.
- 2) ϕ_{JT} : a dilaton/Lagrange multiplier enforcing geometry.

- Teleparallel Gravity in low D

$R \leftrightarrow T$ (Weitzenböck connection which has zero curvature but nonzero torsion)

Instead of using the Ricci scalar R built from the Levi-Civita connection, the basic object is the torsion tensor, which is given in terms of Christoffel symbols $T^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} - \Gamma^\rho_{\mu\nu}$, and the action can be written schematically as

$$S_{TP} = \int d^2x \, e \, T,$$

where T is a scalar constructed from torsion, $T = \frac{1}{4} T^\rho_{\mu\nu} T^\mu_{\rho}{}^{\nu} + \dots$ and $e = \det(e^a_\mu)$ plays the same role as $\sqrt{g}$ in ordinary GR action: invariant volume element from the vielbein. In 2D GR, the action reduces to a topological invariant, so no local dynamics. In 2D teleparallel gravity, the torsion scalar T may also reduce to boundary terms, but depending on the formulation, [it can lead to different topological invariants!](#) Parallel transport around a defect produces not a rotation (as in curvature) but a shift (torsion). Such torsion [dislocations](#) can be localized along lines or points, and one can imagine amplitudes in TP gravity as integrals over configurations of such defects. From this point of view, TP gravity is also topologically trivial in the bulk, but line-like structures (torsion defects, domain walls) carry the physical content.

- Matrix model dual (Triangulations, double scaling limit)

$$Z = \int dM \, e^{-N \text{Tr} V(M)}$$

2D/3D Gravity Approaches

Feature	MLG	JT	Teleparallel Gravity
Action	Liouville action + minimal CFT matter	$S_{JT} = \frac{1}{2} \int d^2x \sqrt{g} \phi (R + 2)$	$S_{TP} = \int d^2x e T$ with torsion scalar T
Geometry control	Liouville mode ϕ (conformal factor)	Dilaton ϕ (enforces $R = -2$)	Torsion $T_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho - \Gamma_{\mu\nu}^\rho$ replaces curvature R
Matter sector	Minimal CFT (finite primaries)	No dynamical matter, only dilaton	Can couple to matter; gravity sector topological in 2D/3D
Degrees of freedom	Coupled matter + Liouville fluctuations	No local DOF (topological)	No local DOF in 2D/3D; only global/topological data
Topology focus	Amplitudes on sphere, torus, higher genus	AdS ₂ geometry, boundary dynamics, Weil-Petersson volumes	Topological invariants (torsion, non-metricity) $\neq$ Euler class
Matrix model relation	Dual to minimal matrix models	Related via double-scaled random matrix ensembles	No direct dual; connections via topological field theory
Physical motivation	Noncritical string theory, solvable 2D gravity	Near-extremal BHs, AdS ₂ /SYK duality	Alternative GR formulation; explores torsion/non-metricity in QG

Comparison Summary

Even though these three version of 2D gravity look quite different, they all share the basic feature that the metric in two dimensions has no propagating degrees of freedom. Instead, what we are really studying are scalar fields that control the geometry — the Liouville mode, the dilaton, or the torsion scalar — and how they encode the quantum or topological aspects of 2D gravity.

Key Questions:

- Topological invariants in 2D gravity. JT vs teleparallel gravity vs Liouville: study the connections.
- Matrix dual discription vs continuum.
- Amplitudes on higher topologies?
- Supersymmetric extensions.

Roadmap of the talk: We can add to the table above another two “directions” g and A_{sym} . We are working with the first column. Our present objective is to study genus dependence and chiral algebra dependence.

$$\{\langle V_{\Delta_1}(z_1) \cdots V_{\Delta_N}(z_N) \rangle_{g, A_{sym}}\} \Rightarrow \text{MLG}(g, A_{sym}) \text{ amplitudes (generating function)}$$

1. Continuous vs. discrete approaches to MLG.
2. Gravitational amplitudes (sphere and torus) via higher equations of motion.
3. Supersymmetric case: Ramond sector, tachyons, physical fields, and the ground ring.

Main points about CFT

- Conserved holomorphic tensor $T(z)$ leads to Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \quad m, n \in \mathbb{Z}$$

- CFT space of local fields $\oplus_{\{\Delta\}}[V_\Delta]$, where $[V_\Delta]$ is highest weight Vir rep-n:

$$L_0|\Delta\rangle = \Delta|\Delta\rangle, \quad L_{n>0}|\Delta\rangle = 0$$

In particular primary fields $V_\Delta(z, \bar{z})$ according to field-state correspondence are related to the highest weight vectors $|\Delta\rangle$

- OPE leads to decomposition of correlation function by means of conformal blocks $F(z)$.

$$\langle V_{\Delta_1}(z_1) \cdots V_{\Delta_N}(z_N) \rangle = \sum_{\text{OPE channels: } \tilde{\Delta}_k} C_{\{\Delta_i\}}^{\{\tilde{\Delta}_k\}} \left| F_{\{\Delta_i\}}^{\{\tilde{\Delta}_k\}}(z) \right|^2$$

- Degenerate conformal fields are relevant for conformal Bootstrap. They constitute spectra of minimal $\mathcal{W}_N$ models, and, in particular, the Virasoro $\text{Vir} = \mathcal{W}_2$ minimal models

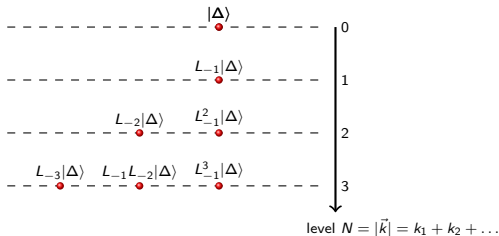
Conformal Bootstrap = degenerate fields + Associativity of operator algebra

Associativity condition for 4-point function $\Rightarrow$ Crossing symmetry.

Virasoro representation theory and conformal blocks

- General Virasoro Verma module: primary fields and descendent fields

$L_{-\vec{k}}|\Delta\rangle = L_{-k_1}L_{-k_2}\dots|\Delta\rangle$, where $k_1 \geq k_2 \geq \dots > 0$.



- Using $c = 1 + 6(b + 1/b)^2$, degenerate representations are labeled by positive integers r, s :

$$\Delta_{r,s} = \frac{(b + 1/b)^2}{4} - \lambda_{rs}^2, \quad \lambda_{rs} = \frac{rb}{2} + \frac{s}{2b}$$

- There exists singular vector $|\chi_{rs}\rangle = \sum_{|\vec{k}|=rs} a_{\vec{k}} L_{-\vec{k}}|\Delta_{rs}\rangle$ in the module $[\Delta_{rs}]$ on the level $r \times s$.

Decoupling condition: setting $|\chi_{r,s}\rangle = 0$ provides an ODE of order rs for any correlation function containing the degenerate field.

Rational and irrational CFTs

Besides the classification by A_{sym} type, there is another principle by which CFT models are divided into two classes:

- Rational – with discrete OPE form ($MM_{q,p}$ models)
- Irrational – with continuous OPE form.

Main example: Liouville theory (or super Liouville)

Super Liouville plays an important role in string theory. The conformal anomaly effect $\Rightarrow$ Liouville theory necessarily arises in all non-critical string models.

Supersymmetric Liouville field theory (SLFT) is CFT (structure constants found from solving Bootstrap by PZ) , which is described by the Lagrangian

$$\mathcal{L}_{\text{SLFT}} = \frac{1}{8\pi} (\partial_a \phi)^2 + \frac{1}{2\pi} (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}) + 2i\mu b^2 \bar{\psi} \psi e^{b\phi} + 2\pi b^2 \mu^2 e^{2b\phi} ,$$

the coupling constant b is related to the central charge $\hat{c} = 1 + 2(b + 1/b)^2$ of the SVIR algebra. Primary fields are exponentials $e^{a\phi}$ (with superpartners).

Torus Virasoro conformal blocks

The partition function for a conformal field theory defined on a torus with modular parameter τ is

$$Z = \text{Tr} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right), \quad \text{where} \quad q = e^{2\pi i \tau}, \quad \text{and} \quad H = L_0 + \bar{L}_0$$

Example: 1-point block in the torus CFT_2 is defined as the holomorphic contribution to the 1-point correlation function of a given primary operator,

$$\langle \mathcal{O}_{\Delta, \Delta}(z, \bar{z}) \rangle = \text{Tr} \left(q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} \mathcal{O}_{\Delta, \Delta}(z, \bar{z}) \right) = \sum_{\tilde{\Delta}} C_{\tilde{\Delta}}^{\Delta} \mathcal{F}_{\tilde{\Delta}, \Delta}(z, q) \mathcal{F}_{\tilde{\Delta}, \Delta}(\bar{z}, \bar{q})$$

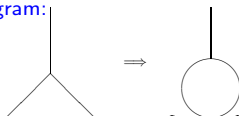
(Holomorphic) 1-point conformal block (does not depend on z). Is given by

$$\mathcal{F}_{\tilde{\Delta}, \Delta}(z, q) = q^{\tilde{\Delta} - c/24} \sum_{n=0}^{\infty} q^n \mathcal{F}_n(\Delta, \tilde{\Delta}, c)$$

q is the elliptic parameter on a torus with the modulus τ , and the expansion coefficients are

$$\mathcal{F}_n(\Delta, \tilde{\Delta}, c) = \frac{1}{\langle \tilde{\Delta} | \mathcal{O}_{\Delta} | \tilde{\Delta} \rangle} \sum_{n=|M|=|N|} B^{M|N} \langle \tilde{\Delta}, M | \mathcal{O}_{\Delta} | N, \tilde{\Delta} \rangle$$

where $|\tilde{\Delta}, M\rangle = L_{-m_1}^{i_1} \dots L_{-m_k}^{i_k} |\tilde{\Delta}\rangle$ are descendant vectors in the Verma module generated from the primary state $|\tilde{\Delta}\rangle$. Here, M labels basis monomials, $|M| = i_1 m_1 + \dots + i_k m_k$ denotes the sum of the Virasoro generator indices. The matrix $B^{M|N}$ is the inverse of the Gram matrix $B_{M|N} = \langle \tilde{\Delta}, M | N, \tilde{\Delta} \rangle$. **Tadpole diagram:**



Exactly solvable models of non-critical strings. MLG

Target space: $MM_{q,p}$. Non-critical string $\Rightarrow$ in action has a contribution of Liouville theory (from the integral over 2d metrics) $\Rightarrow$ interpretation as a model of induced 2d gravity.

MLG structure:

- Action : $A_{LG} = A_{CFT} + A_L + A_{ghosts}$. A_{CFT} is the matter sector (Minimal model), A_L is the gravitational sector, A_{ghosts} is the ghost sector (B_2, C_{-1}).
- MLG - Q BRST theory, $Q = \oint dz [T^M(z) + T^L(z) + \frac{1}{2} T^{gh}(z)] c(z)$
BRST inv condition $\Rightarrow c_{q,p} + c_L = 26$, then $Q^2 = 0$
- Physical fields – Q_{BRST} cohomology: $QW_{mn} = 0$, W_{mn} is not exact.
Explicit form of physical fields $W_{mn} = U_{m,n} c \bar{c}$, where $U_{m,n} = \Phi_{m,n} V_{m,-n}$ and $\Delta(\Phi_{m,n}) + \Delta(V_{m,-n}) = 1$.

Corr. numbers: $\prod_{i=4}^n \int d^2 z_i \langle W_1(0) W_2(1) W_3(\infty) U_i(z_i) \rangle$

The calculation requires integrating over the moduli space of the product of the corr. functions in math. and grav. sectors, use conformal block decomposition, Liouville contour problem, ...

Method for calculating the MLG corr. numbers

The method is based on the relationship between the physical fields of the MLG and the logarithmic fields of the Liouville theory.

The log fields in the Liouville theory $V'_{mn}(z) = \frac{d}{da} V_a(z)|_{a=a_{mn}}$ where $V_a =: e^{a\phi(z)}$:

HQEM: $D_{mn}\bar{D}_{mn}V'_{mn} = B_{mn}V_{m,-n}$, where B_{mn} is a numeral. coefficients,

$V_{m,-n}$ is the dressing field of Liouville in W_{mn} and D_{mn} is the operator, creating singlar vector.

Example: $V'_{11} = \phi \Rightarrow \partial\bar{\partial}\phi = e^\phi$ is the Liouville equation.

Using HQEM $\Rightarrow$ The main relation: $W_{mn} = B_{mn}^{-1}\bar{Q}_B Q_B O'_{mn}$

where $O'_{mn} = H_{mn}\bar{H}_{mn}\Phi_{mn}V'_{mn}$, where H_{mn} is found explicitly.

Applying $b_{-1}\bar{b}_{-1}$ and using $L_n = \{b_n, Q\} \Rightarrow$

$$U_{mn} = B_{mn}^{-1}\partial\bar{\partial}O'_{mn} \mod Q - \text{exact}$$

Using it in the integral over the moduli space of 4pt numbers by Stokes' theorem $\Rightarrow$ integral over the boundary (the contour around singlar points). The boundary contributions are calculated analytically, using OPE of degenerate fields in $O'_{mn} (O'_{mn} W_{m_i n_i} = \dots)$.

Alternative approach to MLG and string equation

In the direct approach, the problem of calculating arbitrary n -point functions and generalizing to higher genera remains open.

The alternative approach is based on the idea of **double scaling limit** which describes a statistical system living on random lattices that fluctuate and are also an object of a statistical ensemble, at a phase transition point and in a situation where large area surfaces contribute.

Such a description leads to a rigorous mathematical formulation of 2D gravity models (Douglas).

The generating function is constructed from some special solution of the string equation

$[P, Q] = 1$, where P, Q are differential operators,

depending on the set of parameters $\{\lambda_{mn}\}$ and $\{u_i\}$.

$$Z(\lambda) = \langle \exp \sum_{mn} \lambda_{mn} W_{mn} \rangle \quad \frac{\partial^2 Z}{\partial \lambda_{11}^2} = u_1^*(\lambda),$$

where $u_1^*(\lambda)$ is a special solution of the string equation. As of yet $(2, 2p + 1)$, i.e. $q = 1!$

String equation and Frobenius manifolds

Using the connection with FM, it was possible to determine the required solution and construct a closed expression for the generating function of the correlation numbers in MLG.

The choice of flat coordinates on the FM plays a key role.

The explicit representation is of the form

$$\langle \exp \sum_{mn} \lambda_{mn} W_{mn} \rangle = \int_0^{v^*(\tilde{\lambda})} C_{\alpha}^{\beta\gamma} \frac{\partial S}{\partial v^{\beta}} \frac{\partial S}{\partial v^{\gamma}} dv^{\alpha}, S = \operatorname{res}_{y=\infty} \sum_{m,n} \tilde{\lambda}_{mn} Q^{\frac{(q+1)m-qn}{q}}$$

where v^{α} - flat coordinates, $\alpha, \beta, \gamma = 1, \dots, q-1$, v^* solution of the string equation $\partial S / \partial v_{\alpha} = 0$, $C_{\alpha}^{\beta\gamma} \Leftarrow \operatorname{FA}_q: \mathbb{C}[y] / \frac{dQ}{dy}$, $Q(y) = \sum_k u_k y^{q-k}$

$$\{C_{\alpha}^{\beta\gamma}, v^*(\lambda)\}$$

$\Downarrow$

$$\langle W_{m_1 n_1} \cdots W_{m_N n_N} \rangle$$

Physical amplitudes

Explicit construction of correlation numbers

- ▷ Relationship of MLG $\leftrightarrow$ FM with FA_q algebra and metric $(e_{\alpha}, e_{\beta}) = \operatorname{res}_{y=\infty} \frac{e_{\alpha} \cdot e_{\beta}}{Q'}$
- ▷ Explicit form of $C_{\alpha}^{\beta\gamma}$, explicit form of the solution of the string equation in the flat coordinates $v_{\alpha}(\lambda_{mn})$.

Some properties of CFT correlators on a torus

τ is the modular parameter of the torus and $q \equiv \exp(2\pi i\tau)$.

Conformal Ward identities on the torus:

$$\langle T(z)\Phi_{\Delta}(x) \rangle = \left[\Delta (\mathcal{P}(z-x) + 2\eta_1) + (\zeta(z-x) + 2\eta_1 x) \partial_x + 2\pi i \frac{\partial}{\partial \tau} \right] \langle \Phi_{\Delta}(x) \rangle$$

Here elliptic ζ -function and Weierstrass $\mathcal{P}$ -function behave as $\zeta(z) \sim 1/z$ and $\mathcal{P}(z) \sim 1/z^2$ at $z = 0$ (in this sense they are doubly periodic analogues of the corresponding terms arising in the conformal Ward identities on the sphere) and

$$\eta_1 = (2\pi)^2 \left[\frac{1}{24} + \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \right] .$$

One-point functions of primary fields $f_{\Delta}(\tau) \equiv \langle \Phi_{\Delta} \rangle_{\tau}$ are modular forms of weight Δ , which means that under transformation $\tau \rightarrow \tau + 1$ they are invariant and under $\tau \rightarrow -\frac{1}{\tau}$ they transform as

$$f_{\Delta}\left(-\frac{1}{\tau}\right) = (\tau\bar{\tau})^{\Delta} f_{\Delta}(\tau)$$

Calculation of torus one-point numbers

The correlator $\langle W_{m,n} \overline{B} \overline{B} \rangle_\tau$ is a modular form of weight $(2, 2)$. It follows that we get a well-defined correlation number if we integrate this correlator over one-punctured torus moduli space, which is a fundamental domain of the $PSL(2, \mathbb{Z})$ action on upper half-plane.

$$\langle \langle W_{1,n} \rangle \rangle_{g=1} = \int_F d^2\tau \langle B \overline{B} C \overline{C} V_{1,-n} \Phi_{1,n} \rangle_\tau$$

Ghost sector:

$$\langle B(z) C(w) \overline{B}(\overline{z}) \overline{C}(\overline{w}) \rangle = |\eta(q)|^4, \quad \eta(q) \equiv q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

Notice independence from the positions of the ghosts.

Minimal model one-point correlators:

$$\langle \Phi_{1,k} \rangle = \sum_{m=1}^p C_{m,(1,k)}^{(M)m} |q|^{2\Delta_{1,m}^M - \frac{1}{12} + \frac{(b^{-1}-b)^2}{2}} |F_M(\Delta_{1,k}^M, \Delta_{1,m}^M, q)|^2.$$

Liouville sector:

$$\langle V_a \rangle_\tau = \int_\gamma \frac{dP}{4\pi} C_{a, Q/2+iP}^{(L)Q/2+iP} (q\overline{q})^{-1/24+P^2} \times |F_L(\Delta_a^L, \Delta_{Q/2+iP}^L, q)|^2,$$

Reduction to boundary terms

One can commute the BRST operator with the remaining B -ghosts using $\{Q, B(z)\} = T(z)$, where $T = T_L + T_M + T_{gh}$ is the stress-energy tensor of the full theory, and discard Q -exact terms.

$$\langle\langle W_{1,k} \rangle\rangle_{g=1} = B_{1,k}^{-1} \int d^2\tau \langle B(z) \bar{B}(\bar{z}) \mathcal{Q} \bar{\mathcal{Q}} (O'_{m,n}) \rangle_\tau = B_{1,k}^{-1} \int d^2\tau \langle T(z) \bar{T}(\bar{z}) O'_{m,n} \rangle_\tau .$$

We find the following general formula

$$\langle\langle W_{1,k} \rangle\rangle_{g=1} = k(2p + 1 - k)$$

Super symmetric extension

The symmetry algebra of SMLG is $\mathcal{N} = 1$ superconformal algebra,

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{\hat{c}}{8}(n^3 - n)\delta_{n,-m},$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{\hat{c}}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s},$$

$$[L_n, G_r] = \left(\frac{1}{2}n - r\right)G_{n+r},$$

where

$$r, s \in \mathbb{Z} + \frac{1}{2} \quad \text{for the NS sector,}$$

$$r, s \in \mathbb{Z} \quad \text{for the R sector.}$$

The SMLG is a tensor product of superconformal matter, super Liouville and superghosts, anticommuting fields (b_2, c_{-1}) and commuting $(\beta_{3/2}, \gamma_{-1/2})$, with $c_{gh} = -10$

$$A_{\text{SLG}} = A_{\text{SM}} + A_{\text{SL}} + A_{\text{SG}}$$

each of which obeys the symmetry with the central charge parameters constrained by

$$\hat{c}_{\text{SM}} + \hat{c}_{\text{SL}} + \hat{c}_{\text{SG}} = 0$$

BRST quantization. NS sector

BRST charge $\mathcal{Q}$ is given by

$$\mathcal{Q} = \frac{1}{2\pi i} \oint dz j_{\mathcal{Q}}(z),$$

$$j_{\mathcal{Q}}(z) =: c(z) \left(T^L(z) + T^M(z) + \frac{1}{2} T^g(z) \right) + \gamma(z) \left(G^L(z) + G^M(z) + \frac{1}{2} G^g(z) \right)$$

NS Physical fields. $L_n = \{Q, b_n\} \Rightarrow \Delta^{tot}(\Psi) = 0$. Indeed, $L_0\Psi = Qb_0\Psi$, so $b_0\Psi = 0$, otherwise it is Q-exact, but then it follows that $L_0\Psi = 0$. In NS sector, there exist two types of physical fields

$$\mathbb{W}_a(z, \bar{z}) = \mathbb{U}_a(z, \bar{z}) \cdot c(z) \bar{c}(\bar{z}) \cdot \delta(\gamma(z)) \delta(\bar{\gamma}(\bar{z})),$$

Here $\delta(\gamma)$ is defined using formal properties of the Dirac δ -function, $\Delta(\gamma) = 1/2$. Second type:

$$\tilde{\mathbb{W}}_a(z, \bar{z}) = \left(\bar{G}_{-1/2}^{M+L} + \frac{1}{2} \bar{G}_{-1/2}^g \right) \left(G_{-1/2}^{M+L} + \frac{1}{2} G_{-1/2}^g \right) \mathbb{U}_a(z, \bar{z}) \cdot \bar{c}(\bar{z}) c(z),$$

where

$$\mathbb{U}_a(z, \bar{z}) = \Phi_{a-b}(z, \bar{z}) V_a(z, \bar{z}).$$

The parameter a can take generic values. The general form of the n -point correlator

$$I_n(a_1, \dots, a_n) = \prod_{i=4}^n \int d^2 z_i \left\langle \bar{G}_{-1/2} G_{-1/2} \mathbb{U}_{a_i}(z_i) \tilde{\mathbb{W}}_{a_1}(z_1) \mathbb{W}_{a_2}(z_2) \mathbb{W}_{a_3}(z_3) \right\rangle.$$

Physical fields in the R sector

Physical fields $|\Psi\rangle$ satisfy the following requirements

$$\mathcal{Q}|\Psi\rangle = 0, \quad |\Psi\rangle \neq \mathcal{Q}[\dots], \quad (1)$$

$$b_0|\Psi\rangle = L_0|\Psi\rangle = 0, \quad \text{and} \quad \beta_0|\Psi\rangle = G_0|\Psi\rangle = 0, \quad (G_n = [Q, \beta_n]) \quad (2)$$

As well as in the NS sector we construct $|\Psi\rangle$ from the primary fields in the Matter and Liouville sector

$$|\mathbb{U}_a^R\rangle = \sum_{\epsilon, \epsilon' = \pm 1} u_{\epsilon, \epsilon'}^R |\Theta_{a-b}^\epsilon\rangle |R_a^{\epsilon'}\rangle. \quad (3)$$

Its dimension is

$$\Delta(\mathbb{U}_a^R) = 5/8. \quad (4)$$

In order to fulfill the condition $b_0|\Psi\rangle = 0$, in the (bc) sector, the state $|\Psi\rangle$ must contain a vacuum $|\nu\rangle_{bc}$ according to

$$\begin{cases} b_n|\nu\rangle_{bc} = 0, & n \geq 0, \\ c_m|\nu\rangle_{bc} = 0, & m \geq 1. \end{cases} \quad (5)$$

The state $|\nu\rangle_{bc}$ corresponds to the field $c(z)$ with conformal dimension $\Delta(c) = -1$.

Ghost number balance rules

The conservation of the ghost current restricts the types of fields that can appear in non-vanishing correlation functions of SMLG for NS sector (e.g. $\langle \widetilde{WWW} \rangle$):

$$N_c - N_b = 3, \quad N_{\delta(\gamma)} - N_{\delta(\beta)} + N_\beta - N_\gamma = 2.$$

OPE for the correlation function with N_c and N_b numbers of c and b fields respectively, we insert the operator $N_{bc}^g = \oint dz J^{bc}$, where $J^{bc} = - : bc :$

$$X = \oint du \langle J^{bc}(u) \dots c(z_1) \dots c(z_{N_c}) b(y_1) \dots b(y_{N_b}) \rangle = (N_c - N_b) \langle \dots c(z_1) \dots c(z_{N_c}) b(y_1) \dots b(y_{N_b}) \rangle$$

deforming the contour to infinity and using the transformation law of the ghost current

$$J^{bc}(u) \rightarrow -J^{bc}(1/u)/u^2 + 3/u, \quad u \rightarrow 1/u, \quad (6)$$

we find that the same integral becomes

$$\begin{aligned} X &= \oint_{\infty} du [\langle (-J(1/u)/u^2 + 3/u) \dots c(1/z_1) \dots c(1/z_{N_c}) b(1/y_1) \dots b(1/y_{N_b}) \rangle] \\ &= 3 \langle \dots c(z_1) \dots c(z_{N_c}) b(y_1) \dots b(y_{N_b}) \rangle. \end{aligned}$$

which proves the first rule.

For the $\beta\gamma$ system, we have the ghost current $J^{\beta\gamma} = - : \beta\gamma := -\partial\phi$ which transforms as

$$J^{\beta\gamma}(u) \rightarrow -J^{\beta\gamma}(1/u)/u^2 - 2/u, \quad u \rightarrow 1/u.$$

Using the OPEs of $J^{\beta\gamma}$ with $\beta, \gamma, \delta(\gamma)$, one arrives at the second rule.

Ghost number balance rules. R sector

For the Ramond sector, it is convenient to work in the bosonized representation, where $\sigma = e^{-\phi/2}$, $\Delta(\sigma) = 3/8$. From which we construct physical field $R = \mathbb{U}\sigma c$. Indeed, for an arbitrary exponential field of the form $e^{I\phi}$, its conformal dimension is given by $-I(I+2)/2$. For our purposes, we introduce also the field $\sigma_2 = e^{\phi/2}$, which has conformal dimension $-5/8$. This allows one to construct fields (satisfying zero total dimension condition) from σ_2 , c , and suitable descendants from the Liouville and matter sectors involving modes such as G_{-1} and L_{-1} . For any exponential field, we have the OPE:

$$J^{\beta\gamma}(u)e^{I\phi(0)} = \frac{I}{u}e^{I\phi(0)}. \quad (7)$$

From this OPE, it follows that the modified ghost balance rule for correlation functions in the Ramond sector, which include insertions of σ and σ_2 , must take the form

$$N_{\delta(\gamma)} - N_{\delta(\beta)} + N_{\beta} - N_{\gamma} + N_{\sigma}/2 - N_{\sigma_2}/2 = 2.$$

Restricting this rule to the case where only the fields $\delta(\gamma)$, σ , and σ_2 appear in the $\beta\gamma$ sector of the correlation function, we get (e.g. $\langle \text{RRW} \rangle$ - yes, and $\langle \text{RR}\tilde{\text{W}} \rangle$ -no, this leads to idea of connection with the ground ring)

$$N_{\delta(\gamma)} + N_{\sigma}/2 - N_{\sigma_2}/2 = 2. \quad (8)$$

In view of the structure of the R physical field, the presence of σ_2 appears to be essential. For instance, if $N_{\delta(\gamma)} = 2$ ($\#W = 2$), one must include pairs of (σ, σ_2) in higher-point functions.

Ground ring operators

Similarly to bosonic case, the Ramond sector also contains “discrete states” $[\sigma] = 3/8$, $[U_{mn}] = 5/8 \Rightarrow [Y_{mn}] = 5/8 - mn/2$

$$\mathbb{O}_{m,n} = \bar{H}_{m,n} H_{m,n} \mathbb{Y}_{m,n}^R \bar{\sigma} \sigma |1\rangle_{bc}, \quad \mathbb{Y}_{m,n}^R = (\Theta_{m,n}^- R_{m,n}^+ + i \Theta_{m,n}^+ R_{m,n}^-), \quad (9)$$

where $|1\rangle_{bc} = b_{-1}|v\rangle_{bc}$ and the operators H_{mn} are operators of dimension $\frac{mn}{2} - 1$ built from super-Virasoro generators. They are fixed by requirement that $\mathbb{O}_{m,n}$ is BRST-closed, and therefore

$$\mathcal{Q} H_{m,n} \mathbb{Y}_{m,n}^R \sigma = \frac{1}{2} \left(1 - \frac{1}{(\beta_{m,n}^M)^2} G_0^L G_0^M \right) D_{m,n}^L \mathbb{Y}_{m,n}^R c \sigma. \quad (10)$$

For $\mathbb{O}_{1,2}$, we find

$$H_{1,2} = \frac{1}{2} + \frac{b^2}{1-2b^2} G_0^M \beta_{-1} c_1 - \frac{b^2}{1+2b^2} G_0^L \beta_{-1} c_1 + \frac{4b^2}{(1-2b^2)(1+2b^2)} G_0^L G_0^M. \quad (11)$$

Using Higher Equations of Motion and the fact that $|\mathbb{U}_{m,n}\rangle$ is an eigenstate of the product $G_0^L G_0^M$

$$G_0^L G_0^M \bar{D}_{m,n}^L D_{m,n}^L (\mathbb{Y}_{m,n}^R)' = B_{m,n} G_0^L G_0^M \mathbb{U}_{m,-n}^R = i B_{m,n} \beta_{m,n}^M \beta_{m,-n}^L \mathbb{U}_{m,-n}^R, \quad (12)$$

we derive the following representation for the physical fields

$$\bar{\mathcal{Q}} \mathcal{Q} \mathbb{O}'_{m,n} = B_{m,n} \mathbb{R}_{m,-n}. \quad (13)$$

Acting on both sides of equation (13) with $\bar{b}_{-1} b_{-1}$ we obtain the following relation

$$B_{m,n} \mathbb{U}_{m,-n}^R \bar{\sigma} \sigma = (\bar{\partial} - \bar{\mathcal{Q}} \bar{b}_{-1}) (\partial - \mathcal{Q} b_{-1}) \mathbb{O}'_{m,n} = \bar{\partial} \partial \mathbb{O}'_{m,n} + \text{BRST exact terms} \quad (14)$$

which can be used to facilitate evaluation of moduli integrals in four-point correlation numbers.

NS three-point correlation number from ground ring

$$\begin{aligned}
 I_3(a_1, a_2, a_3) &= \langle \tilde{\mathbb{W}}_{m,-n}(z_1) \mathbb{W}_{a_2}(z_2) \mathbb{W}_{a_3}(z_3) \rangle \\
 &= \frac{1}{B_{m,n}} \frac{1}{(2\pi i)^2} \oint_{z_1} d^2 z \langle \bar{j}_{\mathcal{Q}}(\bar{z}) j_{\mathcal{Q}}(z) \mathbb{O}'_{m,n}(z_1) \mathbb{W}_{a_2}(z_2) \mathbb{W}_{a_3}(z_3) \rangle.
 \end{aligned} \tag{15}$$

Thus, we want to compute

$$I_3 = \frac{1}{B_{1,3}} \frac{1}{2\pi i} \oint_{z_1} dz \frac{1}{2\pi i} \oint_{\bar{z}_1} d\bar{z} \langle \bar{j}_{\mathcal{Q}}(\bar{z}) j_{\mathcal{Q}}(z) \mathbb{O}'_{1,3}(z_1) \mathbb{W}_a(z_2) \mathbb{W}_a(z_3) \rangle. \tag{16}$$

Recall that

$$\begin{aligned}
 \mathbb{O}_{13}(x) &= \Phi'_{13}(x) V_{13}(x) - \Phi_{13}(x) V'_{13}(x) + \\
 &\quad \text{term contributing to } \mathbb{W}_a \text{ in the OPE } \mathbb{O}_{1,3} \mathbb{W}_a \\
 &\quad \underbrace{-\Psi_{13}(x) \Lambda_{13}(x)}_{\text{}} + \\
 &\quad + [b^2 : \beta(x) \gamma(x) : + 2b^2 : b(x) c(x) :] \Phi_{13}(x) V_{13}(x) \\
 &\quad - b^2 \beta(x) c(x) \Psi_{13}(x) V_{13}(x) - b^2 \beta(x) c(x) \Phi_{13}(x) \Lambda_{13}(x),
 \end{aligned} \tag{17}$$

where $\Lambda_{13} = G_{-1/2}^L V_{13}$, $\Psi_{13} = G_{-1/2}^M \Phi_{13}$. In the OPE $\mathbb{O}_{1,3} \mathbb{W}_a$ the term proportional to $\mathbb{W}_a$ arises from the third term of the expansion above. Thus we have

$$\mathbb{O}_{1,3}(z_1) \mathbb{W}_a(z_2) = \tilde{\mathbf{C}}_0^M(a-b) \tilde{\mathbf{C}}_0^L(a) \mathbb{W}_a(z_2) + \dots, \tag{18}$$

where $\tilde{\mathbf{C}}_0^M(a-b)$, $\tilde{\mathbf{C}}_0^L(a)$ are special matter and Liouville structure constants given by

$$\tilde{\mathcal{C}}_0^L(a) = \frac{2\pi i \mu \gamma (ab - b^2)}{\gamma(-b^2) \gamma(ab)}, \quad (19)$$

$$\tilde{\mathcal{C}}_0^M(a) = ib^{-2} \gamma \left(\frac{bQ}{2} \right) \left(\frac{\gamma(1-b^2) \gamma(b^2/2 - 1/2)}{\gamma(b^2 - 1) \gamma(3b^2/2 - 1/2)} \right)^{1/2} \frac{\gamma(ab + b^2)}{\gamma(ab)}. \quad (20)$$

Since $V_a(z) = e^{a\varphi(z)}$, hence $V'_a = \partial_a V_a(z) = \varphi(z) V_a(z)$. Thus

$$\mathbb{O}'_{1,3}(z_1) \mathbb{W}_a(z_2) = \varphi(z_1) \left(\tilde{\mathcal{C}}_0^M(a-b) \tilde{\mathcal{C}}_0^L(a) \mathbb{W}_a(z_2) \right). \quad (21)$$

Inserting this OPE into (16) gives

$$I_3 = \frac{\tilde{\mathcal{C}}_0^M(a-b) \tilde{\mathcal{C}}_0^L(a)}{B_{1,3}} \frac{1}{2\pi i} \oint_{z_1} dz \frac{1}{2\pi i} \oint_{\bar{z}_1} d\bar{z} \langle \bar{j}_Q(\bar{z}) j_Q(z) \varphi(z_1) \mathbb{W}_a(z_2) \mathbb{W}_a(z_3) \rangle. \quad (22)$$

The action of j_Q on $\mathbb{W}_a$ gives no contribution since $\mathbb{W}_a$ is physical. The non-zero contribution comes from the OPE $j_Q(z) \varphi(z_1)$. Thus, we have

$$j_Q(z) \varphi(z_1) = (c(z) T(z) + \dots) \varphi(z_1) = c(z) \frac{\varphi'(z)}{z - z_1}. \quad (23)$$

Therefore

$$I_3 = \frac{\tilde{\mathcal{C}}_0^M(a-b) \tilde{\mathcal{C}}_0^L(a)}{B_{1,3}} \frac{1}{2\pi i} \oint_{z_1} dz \frac{1}{2\pi i} \oint_{\bar{z}_1} d\bar{z} \frac{1}{|z - z_1|^2} \langle \bar{c}(\bar{z}) c(z) \partial_z \partial_{\bar{z}} \varphi(z) \mathbb{W}_a(z_2) \mathbb{W}_a(z_3) \rangle. \quad (24)$$

Conclusions

We present a method for the calculation of one-point amplitudes in $(2, 2p + 1)$ minimal Liouville gravity defined on a torus. The method is based on the higher equations of motion in the Liouville CFT. We show that this approach allows to reduce the moduli integrals entering the definition of the torus amplitudes to certain boundary contributions, which can be calculated explicitly.

We verify that the results agree with the calculations performed in the matrix models approach.

We consider the construction of physical fields in the Ramond sector of super minimal Liouville gravity. We derive the ground ring operators and a general relation, that can be used for analytical computations in SMLG. The result is verified on the level of the three-point amplitude.

Perspectives:

- To calculate the general one-loop correlators in $\text{MLG}(p,q)$ using HEM approach
- To construct dual matrix model description for SMLG
- To extend the method based on (S)HEM to multi-point correlators in different topologies