The physics of non-toric simple flops

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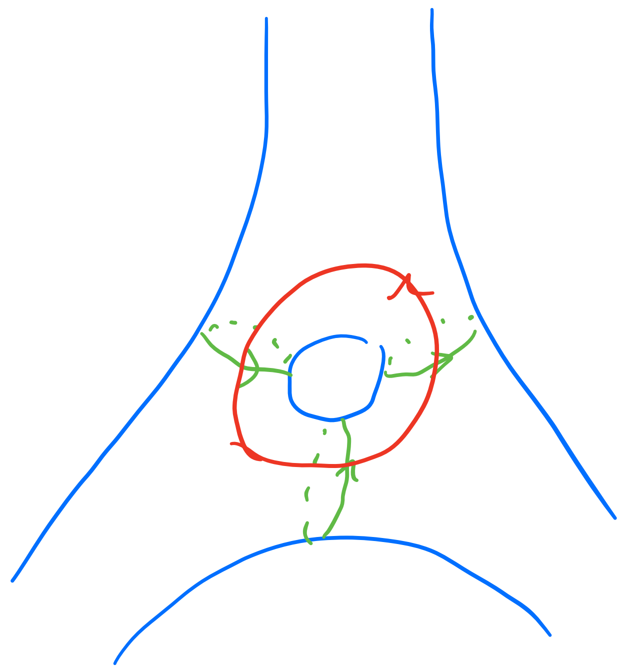
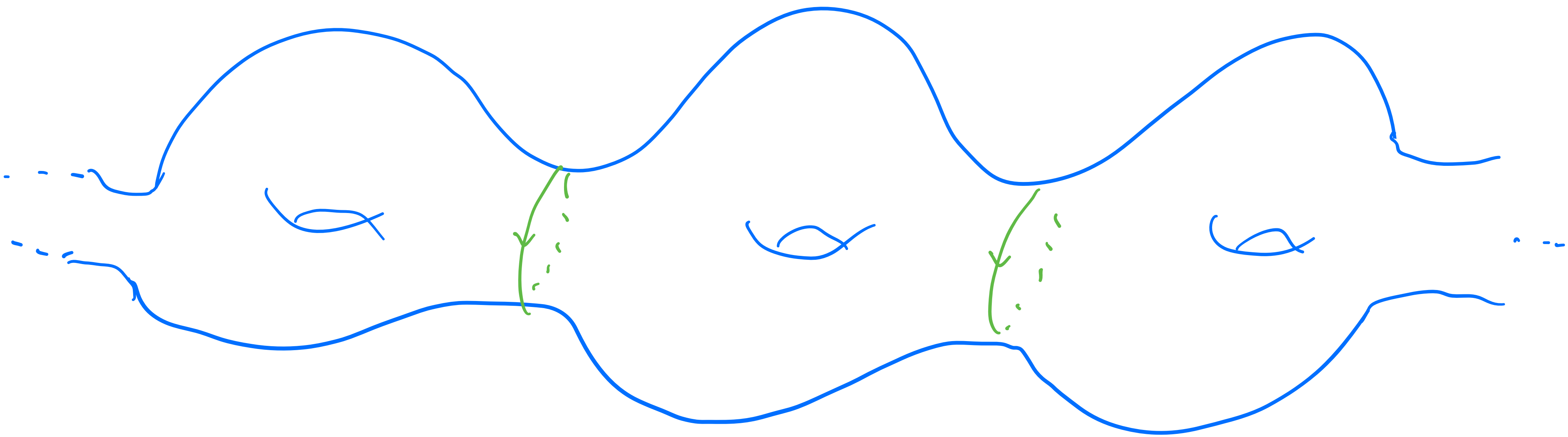
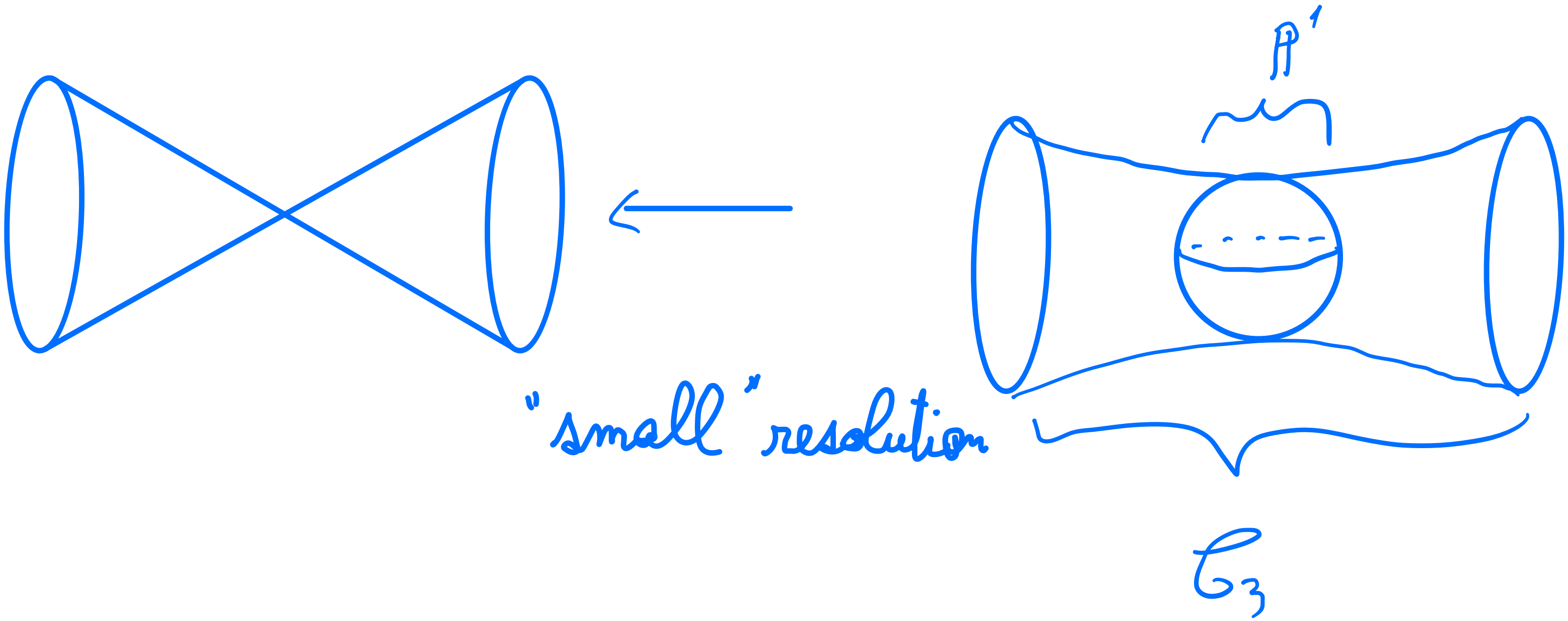
Counting curves in CY 3-folds

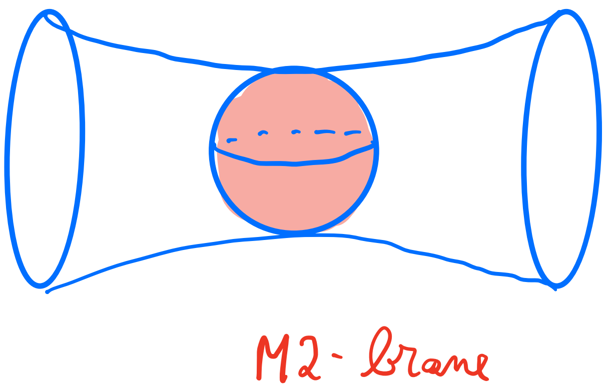




Resolved conifold



$$H_2(\mathcal{C}_3) = \mathbb{Z}\langle \beta \rangle \quad \longleftrightarrow \quad n_\beta^{g=0} = 1$$



$=$ 1 free hyper in 5d $\mathcal{N}=1$.

"Curve counting"

Gromov-Witten:

counting holomorphic maps $\phi : \Sigma_2 \longrightarrow X_3$;

$$Z_{GW}(t, g_s) = \sum_{g \geq 0} \sum_{d} N_g^d \, e^{-dt}$$

t = Kähler parameter

GW-invariants N_g^d :

- virtual dimensions
- can be fractional
- unclear physical interpretation

Gopakumar - Vafa

Counting "images" $\phi(\Sigma_2)$

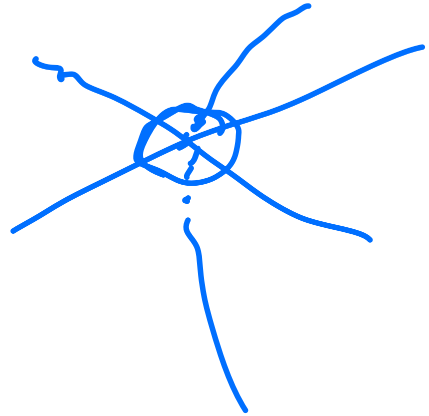
$F_{GW}(t; g_s)$

$$F_{GV}(t, g_s) = \sum_{w=1}^{\infty} \sum_{g=0}^{\infty} \sum_{d} \frac{n_d^g}{w} \left(2 \sinh\left(\frac{w g_s}{2}\right)\right)^{2g-2} e^{-wdt}$$

n_d^g = index of BPS states, physical, integral

In 5d EFT, $n_d^{g=0}$ = # hypers of charge d under $U(1)$ group

Conifold



- There is a $U(1)$ group acting on states:

SUGRA C_3-form $\longrightarrow$ $C_3 = \underbrace{A_{(1)}}_{7d} \wedge \underbrace{\omega_{(2)}}_{\text{normalizable}}$

$$\int \omega_{(2)} \wedge * \omega_{(2)} < \infty$$

$\mathbb{C}^*$-fibers

$$\omega_{(2)} \xleftrightarrow{\text{P.D.}} \text{Divisor class group}$$

- $$Z_{GV} = \sum_{w=1}^{\infty} \sum_{g=0}^{\infty} \frac{1}{w} \left(2 \sinh\left(\frac{w g_s}{2} \right) \right)^{2g-2} e^{-wdt}$$

$\Longrightarrow$ only GV-invariant is $\boxed{n_\beta^{g=0} = 1}$

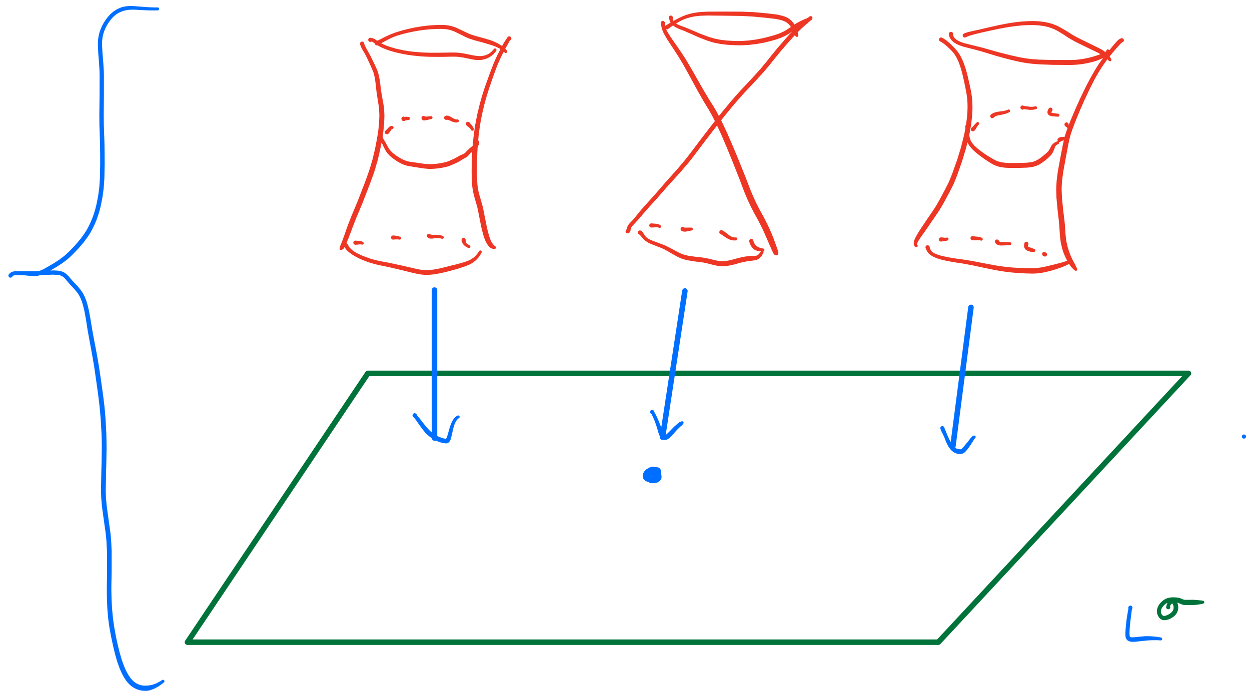
for $H_2(X_3) = \langle \beta \rangle$

Simultaneous resolution

Look @ family of $\mathbb{C}^2/\Gamma_{ADE}$, $\Gamma_{ADE} \subset SL_2\mathbb{C}$

$\boxed{\text{e.g. } \mathbb{C}^2/\mathbb{Z}_2}$ Γ_{a_1} $\sigma \in \mathfrak{T}_{a_1}/W \cong \mathbb{C}/\mathbb{Z}_2$

$uv = z^2 + \sigma$



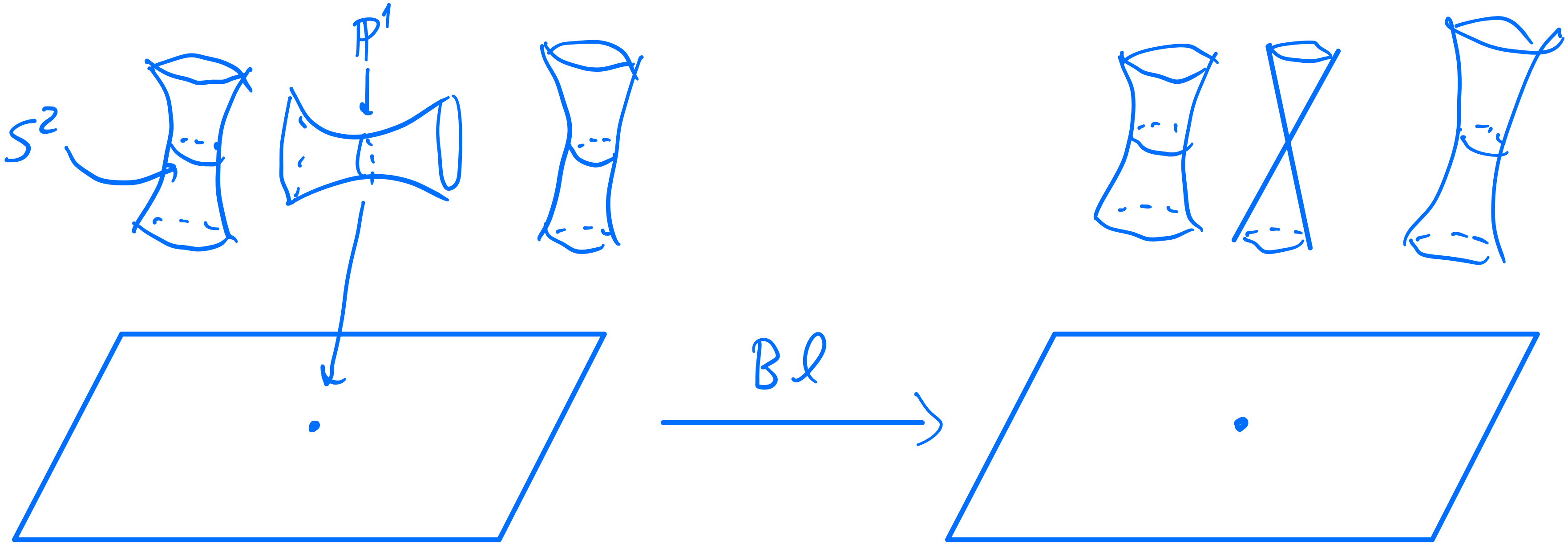
Family is a smooth 3-fold

Base change

$$\begin{tikzcd}
\mathcal{C}_3 = \pi^* X_3 \arrow[rr] \arrow[d, "\pi_1"'] & & X_3 \arrow[d, "\pi_0"] \\
T \arrow[rr] & & T/w \\[-20pt]
t \arrow[rr, mapsto] & & \sigma := t^2
\end{tikzcd}$$

$$\boxed{uv = z^2 + t^2 \longmapsto uv = z^2 + \sigma}$$

Simultaneous resolution



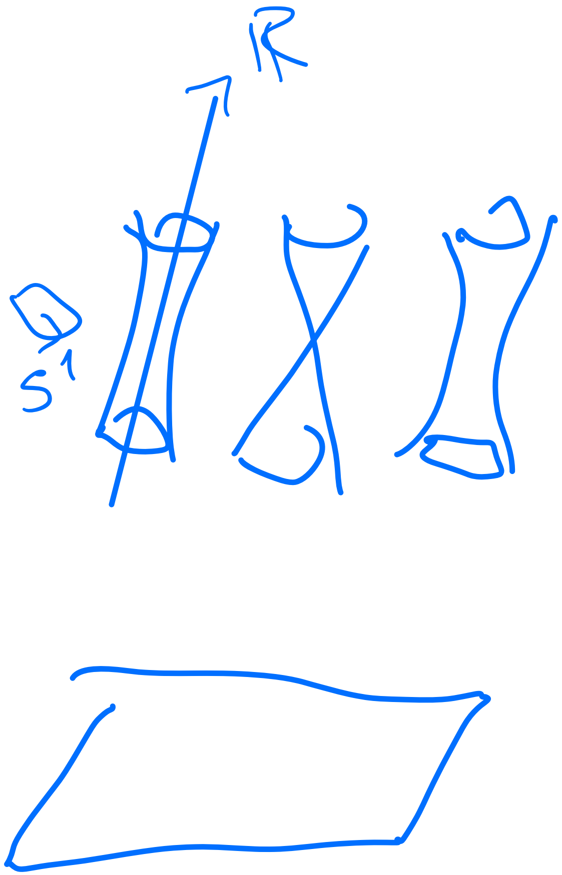
$$\tilde{X}_3 \xrightarrow[\text{small resolution}]{} X_3$$

$$\tilde{X}_3 := \left\{ \begin{pmatrix} u & z+t \\ z-t & v \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = 0 \subset \mathbb{C}^4 \times \mathbb{P}^1 \right\}$$

M-theory $\longrightarrow$ IIA

$CY_3 \circlearrowleft \mathbb{C}^*$

$$\text{M-th} \,/\, CY_3 \xrightarrow{\quad S \quad} \text{IIA on } K3 \times \mathbb{R} + \text{D6-branes}$$



$$\begin{array}{ccc} \mathbb{C}^* & \hookrightarrow & X_3 \\ & & \downarrow \\ & & X_2 \end{array}$$

$\pi : (u, v, z, t) \rightarrow (z, t)$

M-theory $\longrightarrow$ IIA

$CY_3 \circlearrowleft \mathbb{C}^*$

$$uv = z^2 - t^2$$

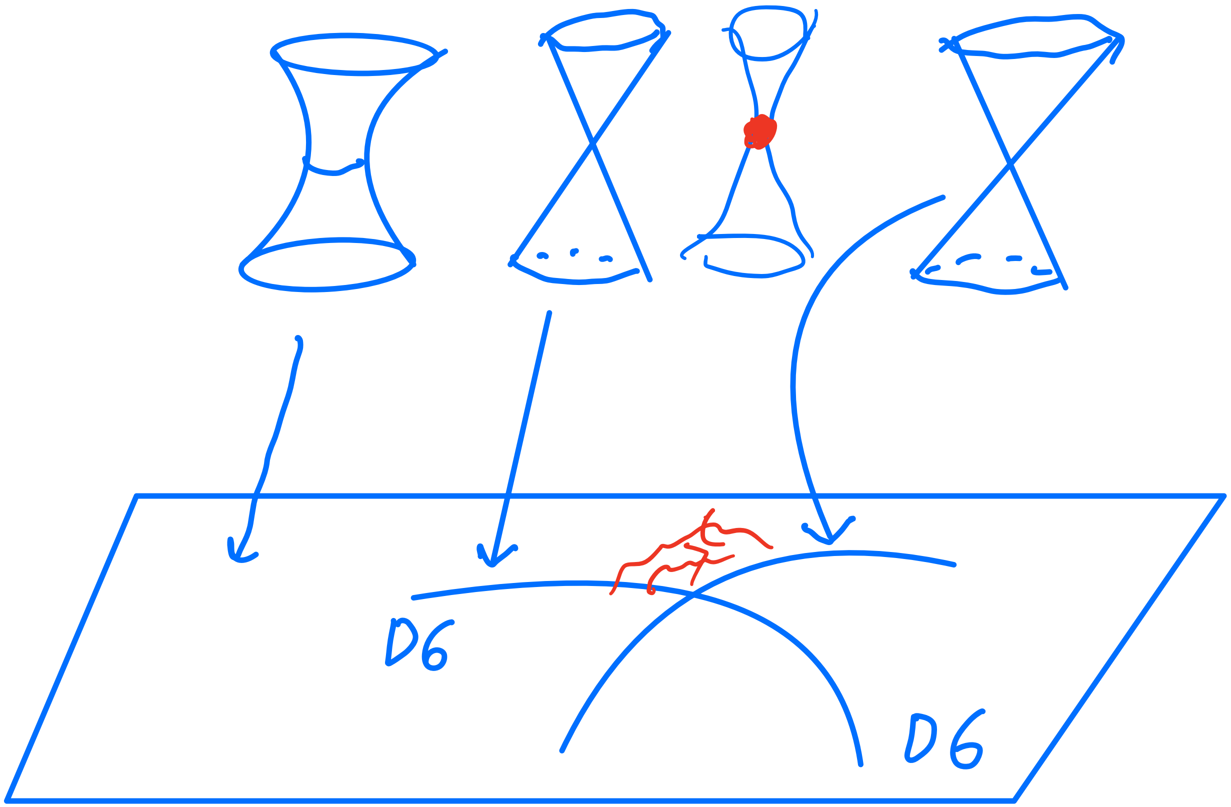
$$uv = \Delta(z,t)$$

M-th / CY_3

$\downarrow$

IIA on K3

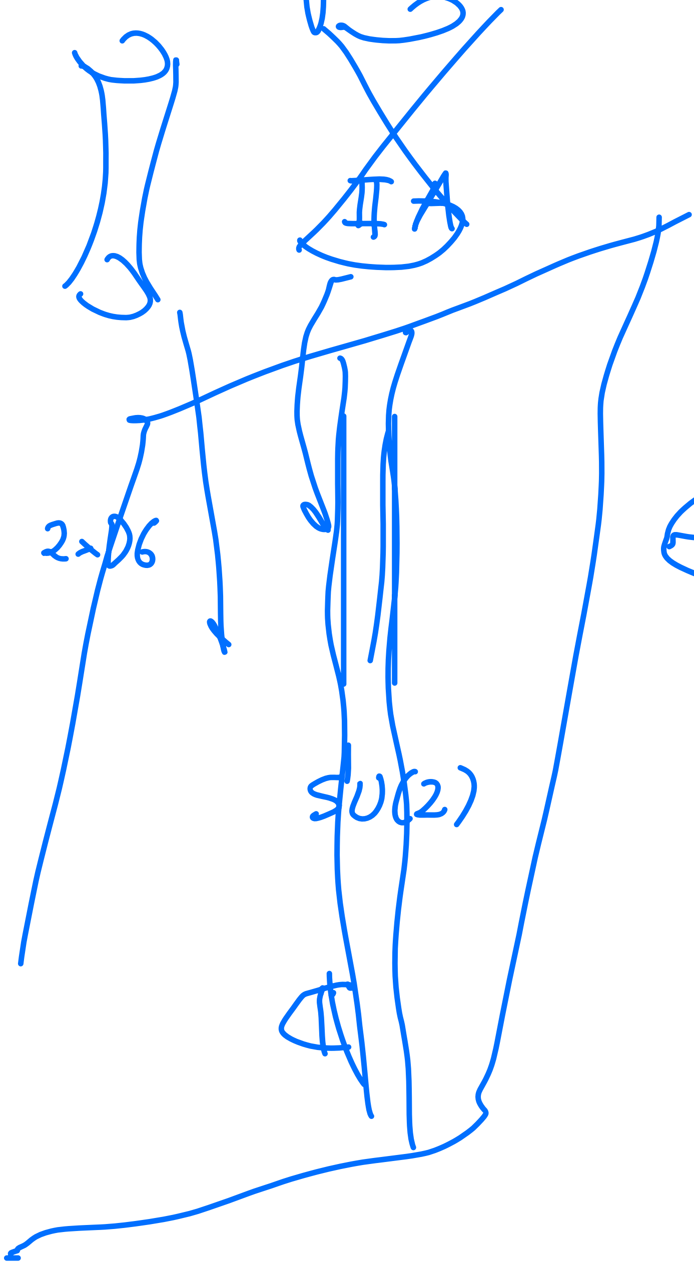
+ D6-branes



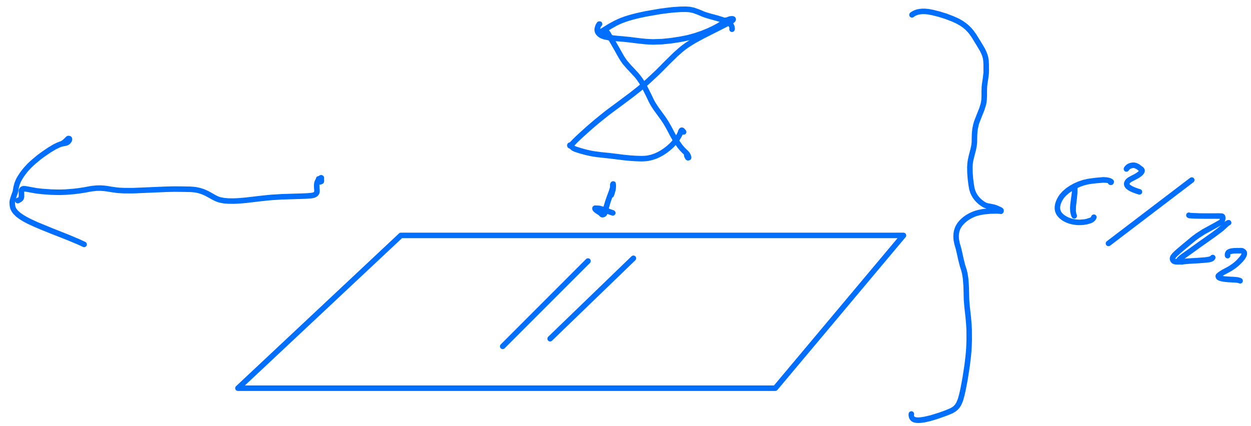
K3

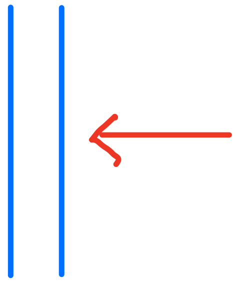
Flops : conifold

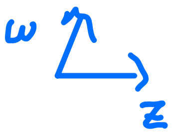
conifold as a family of A_1- surfaces



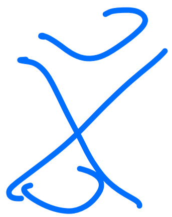
M- theory

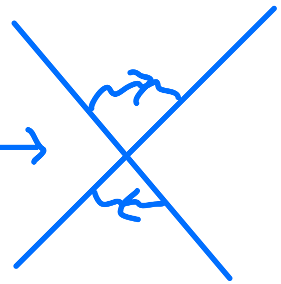


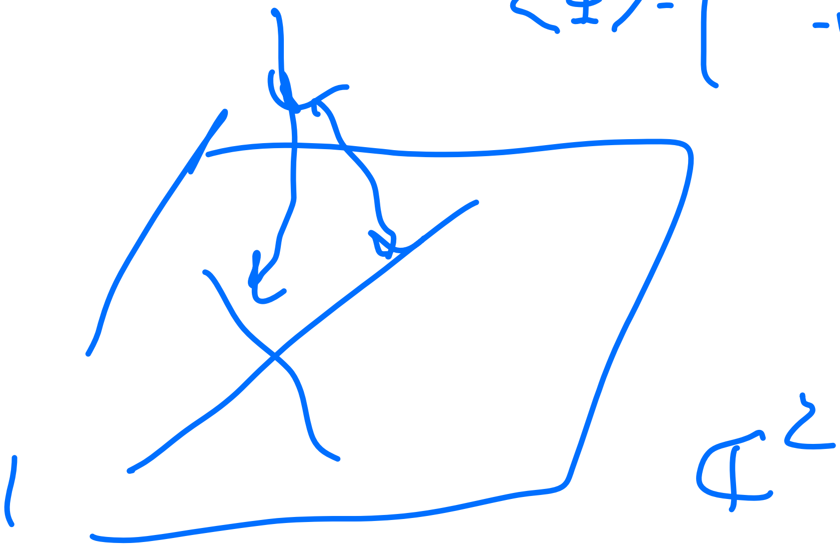
2×D6  7d $SU(2)$-SYM v.m. $\ni (\Phi_1, \varphi_3)$



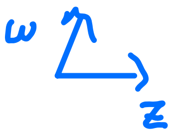
Worldvolume on $z=0$: 7d $\mathcal{N}=1$



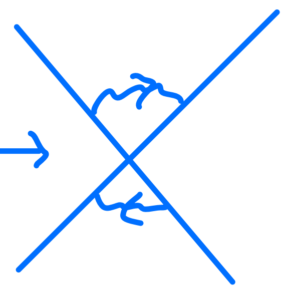
$$\langle \Phi \rangle = \begin{pmatrix} w & \\ & -w \end{pmatrix} \longleftrightarrow$$ 



$2 \times D6$  $\longleftarrow$ 7d $SU(2)$-SYM v.m. $\ni (\Phi_1, \varphi_3)$

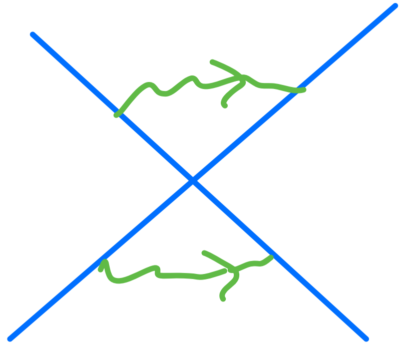


Worldvolume $\qquad\qquad\qquad\qquad$ M-theory

$$\langle \Phi \rangle = \begin{pmatrix} w & \\ & -w \end{pmatrix} \longleftrightarrow$$  $$\longleftarrow \quad uv = \det\left(\mathbb{1}_2 z - \langle \Phi \rangle\right)$$

$$\boxed{uv = z^2 - w^2}$$

7d $\mathcal{N}=1$ $\longrightarrow$ 5d $\mathcal{N}=1$ hyper



$$\delta\Phi \sim \delta\Phi + [\langle\Phi\rangle, x]$$

$$\delta\Phi = \begin{pmatrix}\phi_0 & \phi_+\\ \phi_- & -\phi_0\end{pmatrix} \sim \begin{pmatrix}\phi_0 & \phi_+\\ \phi_- & -\phi_0\end{pmatrix} + 2w\begin{pmatrix}0 & x_+\\ x_- & 0\end{pmatrix}$$

$\Longrightarrow$ ϕ_0 is a 7d mode $\qquad \phi_\pm \in \mathbb{C}[w]/(w)$

$\underbrace{(\phi_+, \phi_-)}_{1\times\ 5d\ \mathcal{N}=1\ \text{hyper}}$ lives @ $w=0$ 5d locus

$1\times$ 5d $\mathcal{N}=1$ hyper $\longleftrightarrow$ $\boxed{n_{GV\,1}^{g=0} = 1}$

Base change II :

e.g. a_2 $\qquad$ $T(a_2) \cong \mathbb{C}^2$

$W_2 = \{1\} \subset W_1 = S_2 \subset W_0 = S_3$

Base change II : e.g. a_2

$\mathfrak{T}(a_2) \cong \mathbb{C}^2$

$W_2 = \{1\}$

$W_1 = S_2$

$W_0 = S_3$

$$\begin{tikzcd}
\mathfrak{T}/W_2 \arrow[r, "\uparrow_2"] \arrow[d] & \mathfrak{T}/W_1 \arrow[r, "\uparrow_1"] \arrow[d] & \mathfrak{T}/W_0 \arrow[d] \\
\mu v = \prod_{i=1}^{3} (z - t_i) & \mu v = (z-\rho_1)(z^2 + \rho_1 z + \rho_2) & \mu v = z^3 + z\sigma_2 + \sigma_3
\end{tikzcd}$$

$\sum_i t_i = 0$

Reid's pagoda

$\underline{a_3}$: Take family of a_3-surfaces such that

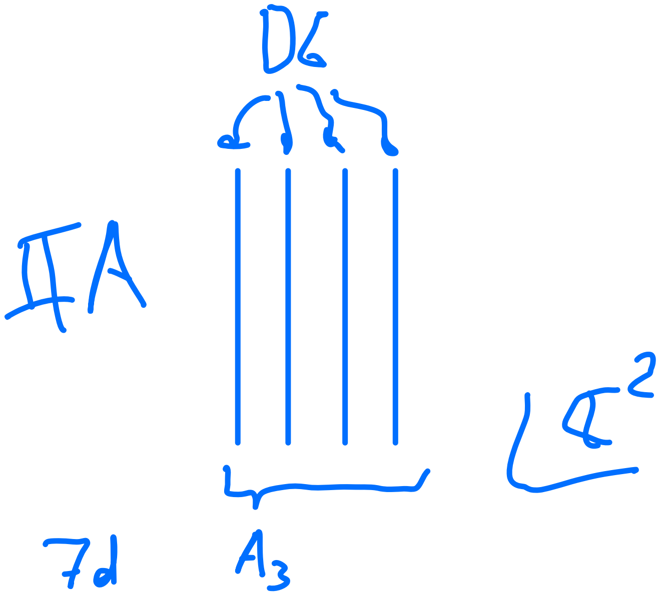
$W = a_1 \times a_1$

$$uv = z^4 - w^2$$

Is a simple flop.

$$N_{\mathbb{P}^1} = \mathcal{O}(0) \oplus \mathcal{O}(-2)_{\mathbb{P}^1}$$

Reid's pagoda : an A_{2k-1}-family



$$\langle \Phi \rangle = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \end{pmatrix}$$

$$\langle \Phi \rangle : A_3 \longrightarrow A_1$$

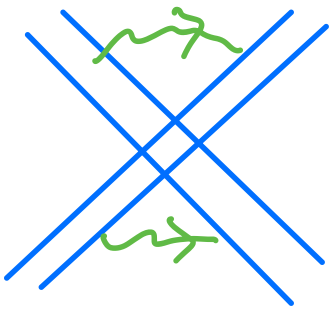
$$X_3 : \quad uv = \det\left(\mathbb{1}_4 z - \langle \Phi \rangle \right)$$

$$uv = z^4$$

$$\mathbb{C}^2 / \mathbb{Z}_4$$

a "T-brane"

next step : $\langle \Phi \rangle = \begin{pmatrix} 0 & 1 & & \\ w & 0 & & \\ & & 0 & 1 \\ & & -w & 0 \end{pmatrix} \sim \begin{pmatrix} \sqrt{w} & & & \\ & -\sqrt{w} & & \\ & & -\sqrt{w} & \\ & & & \sqrt{w} \end{pmatrix}$



$$A_3 \longrightarrow A_1 \longrightarrow U(1)$$

$$\boxed{uv = (z^2 - w)(z^2 + w)}$$

5d hyper

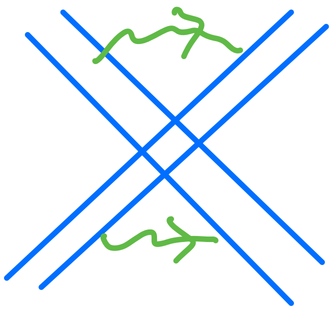
$$\delta\Phi_+ \sim \delta\Phi_+ + [\langle\Phi\rangle, \varepsilon]$$

$$\delta\Phi_+ = \begin{pmatrix} 0 & \begin{matrix} a & 0 \\ b & a \end{matrix} \\ 0 & 0 \end{pmatrix}$$

$a, b \in \mathbb{C}[w]/(w)$

$c \in \mathbb{C}[w]/(w^2)$

next step : $\langle \Phi \rangle = \begin{pmatrix} 0 & 1 & & \\ w & 0 & & \\ & & 0 & 1 \\ & & -w & 0 \end{pmatrix} \sim \begin{pmatrix} \sqrt{w} & & & \\ & -\sqrt{w} & & \\ & & -\sqrt{w} & \\ & & & \sqrt{w} \end{pmatrix}$



$A_3 \longrightarrow A_1 \longrightarrow U(1)$

$$\boxed{uv = (z^2 - w)(z^2 + w)}$$

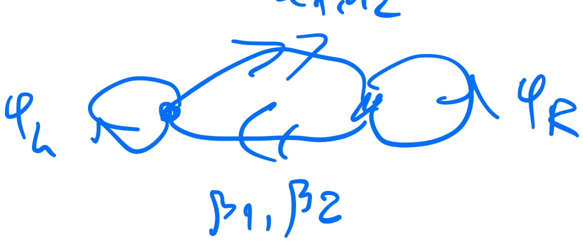
5d hyper

$$\delta\Phi_+ = \begin{pmatrix} 0 & \begin{matrix} a & 0 \\ b & a \end{matrix} \\ 0 & 0 \end{pmatrix} \; ; \; \delta\Phi_-$$

$a, b \in \mathbb{C}[w]/(w)$ $\longleftarrow$ 2 hypers

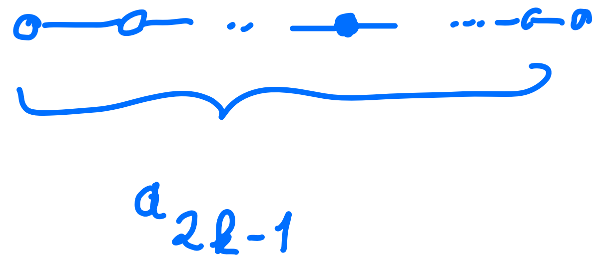
$$\boxed{n_{GV, g=0}^{d=1} = 2}$$

d_1, d_2



k-th Pagoda

$$\langle \Phi \rangle = \begin{pmatrix} \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ w & & & 0 \end{pmatrix} & \\ & \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -w & & & 0 \end{pmatrix} \end{pmatrix}$$



$$uv = z^{2k} - w^2$$

simple flop, $\mathcal{O}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$

$$\boxed{n_{GV,\,g=0}^{d=1} = k}$$

non-toric!

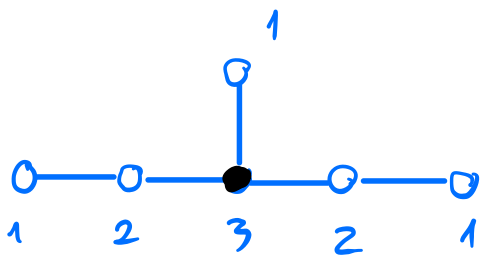
$$W = \varphi_L^{k+1} - \varphi_R^{k+1} + (\varphi_L - \varphi_R)(\alpha_1\beta_1 - \alpha_2\beta_2)$$

General ADE

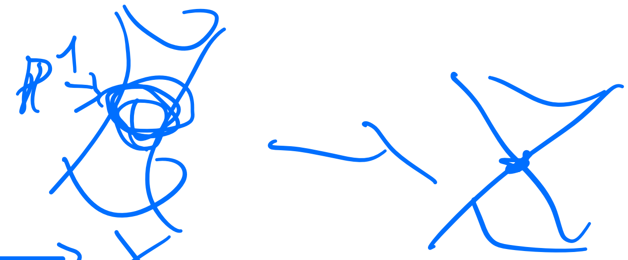
Higher "length":

E.g. Γ_{E_6}

$\mathbb{C}^2/\Gamma_{ADE}$



$\pi : \tilde{X}_3 \xrightarrow[\text{small}]{} X_3$



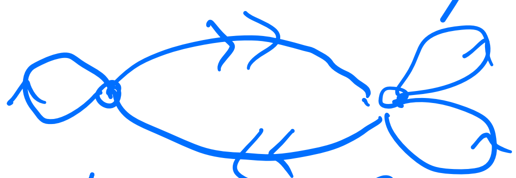
M. WEMYSS

J. KARMAZYM

$\pi^* \mathcal{O}_p = \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}$ length = 3

NCCR:

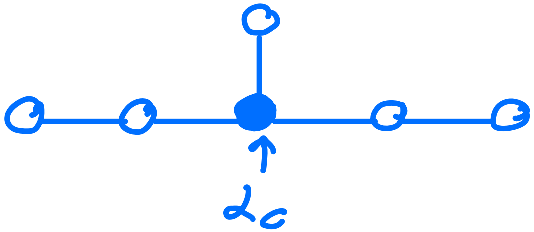
$U(1) \times U(3)$



$\mathbb{C}^1$ $\mathbb{C}^3$

bound states of 3 × M2-branes

ADE - family

E_6 

$\alpha_c^\vee \in \mathfrak{h} \in \mathfrak{e}_6$

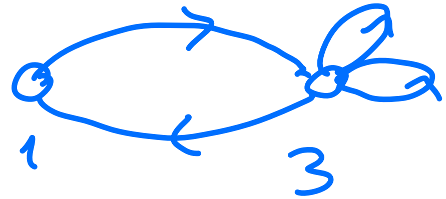
$$\mathrm{Levi}(\alpha_c^\vee) = \text{commutant of } \alpha_c^\vee \subset \mathfrak{e}_6$$

$$= \mathfrak{a}_2^{(1)} \oplus \mathfrak{a}_2^{(2)} \oplus \mathfrak{a}_1 \oplus \langle \alpha_c^\vee \rangle$$

$$\Phi_{\mathfrak{a}_2^{(1)}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ w & 0 & 0 \end{pmatrix}; \quad \Phi_{\mathfrak{a}_2^{(2)}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -w & 0 & 0 \end{pmatrix}; \quad \Phi_{\mathfrak{a}_1} = \begin{pmatrix} 0 & 1 \\ -3w & 0 \end{pmatrix}$$

$$x^2+y^3+z^4+\frac{27w^6}{32}+18w^5+\left(12w^3-\frac{27w^4}{16}\right)y+2\left(w^2-\frac{9w^3}{8}\right)z^2+3wyz^2=0\,.$$

flop of length 3 :



BPS quiver

$$\mathcal{N} = \mathcal{O}(-3) \oplus \mathcal{O}(1)_{\mathbb{P}^1}$$

$$\mathbb{C}[\varepsilon]/(\varepsilon^3)$$

has BPS states of charges up to 3

$$\delta \Phi \sim \delta \phi + [\langle \Phi \rangle, \chi]$$

$$\boxed{n_1^{g=0} = 6 \; ; \quad n_2^{g=0} = 3 \; ; \quad n_3^{g=0} = 1}$$

$\ddots$

E_7, E_8

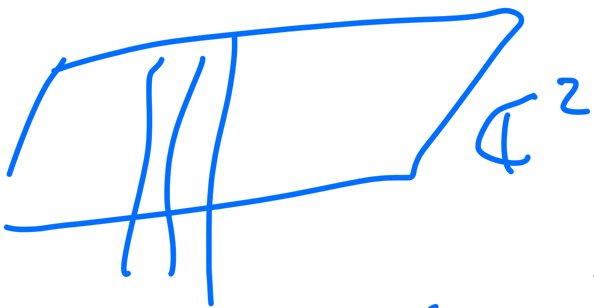
Terminal singularities

$uv = z^3 + w^2 \Longleftarrow$ doesn't admit crepant res

$\Updownarrow$

no NCCR

$\hookrightarrow$ Go to IIA:



$$\langle \underset{\sim}{\Phi} \rangle = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ w & 0 & 0 \end{pmatrix}$$

+ A_n-surface

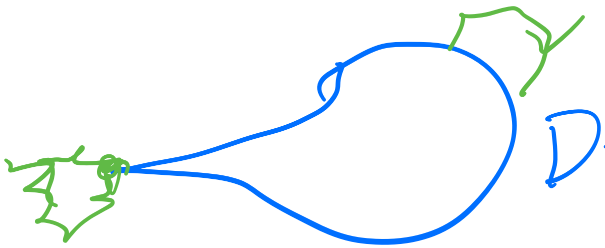
compute

$$\delta\Phi \sim \delta\Phi + [\langle\Phi\rangle, \chi]$$

Terminal singularities

$uv = z^3 + w^2 \Longleftarrow$ doesn't admit crepant res

$\Updownarrow$



D6

no NCCR

$\hookrightarrow$ Go to IIA:

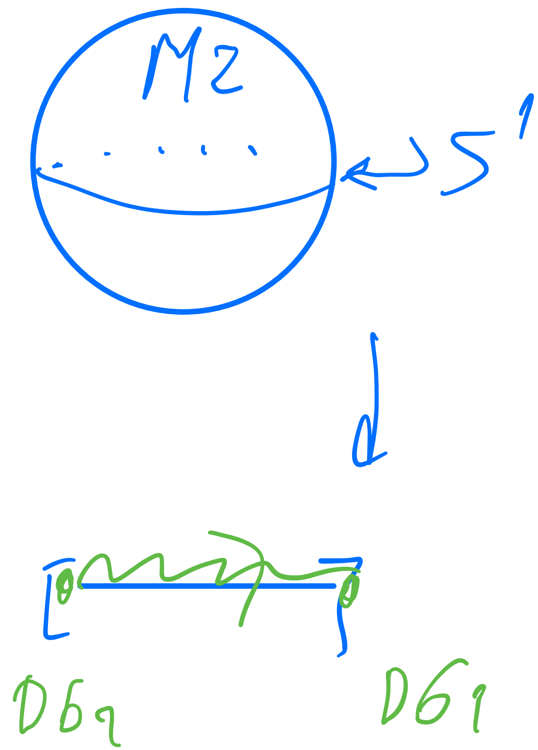
$$\Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ w & 0 & 0 \end{pmatrix}$$

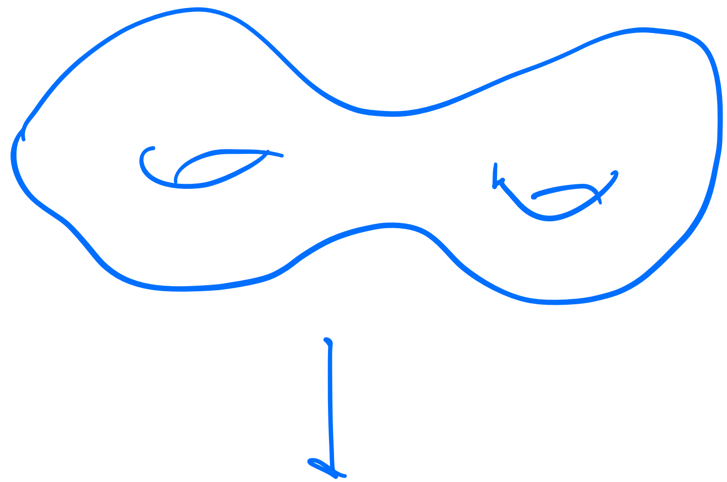
$\rightsquigarrow$ 2 zero-modes $\rightsquigarrow$ 1 hyper

$$\boxed{n_0 \overset{?}{=} 1}$$

Open questions

- Non-simple flops
- Physical interpretation of "width"
- "degree-zero" GV invariants for terminal singularities
- Compact 4-cycles
- Orbifolds and stacks.


M2
S1
D6q
D6q



Parabolic Subalgebras in $\mathfrak{sl}_4$

Θ	$\mathfrak{p}_\Theta$	$\mathfrak{l}_\Theta$	$\mathfrak{n}_\Theta$
○—○—○	$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$
⊛—○—○	$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$
○—⊛—○	$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$
○—○—⊛	$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$	$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$	$\begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
⊛—⊛—○	$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$
○—⊛—⊛	$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$	$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$	$\begin{pmatrix} 0 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
⊛—○—⊛	$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$	$\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
⊛—⊛—⊛	$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$	$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$