A Journey in Resurgence

Veronica Fantini

Université Paris Saclay

Cosmology and High Energy Physics
September 18 2025

Paroisse Sainte Bernadette, Montpellier

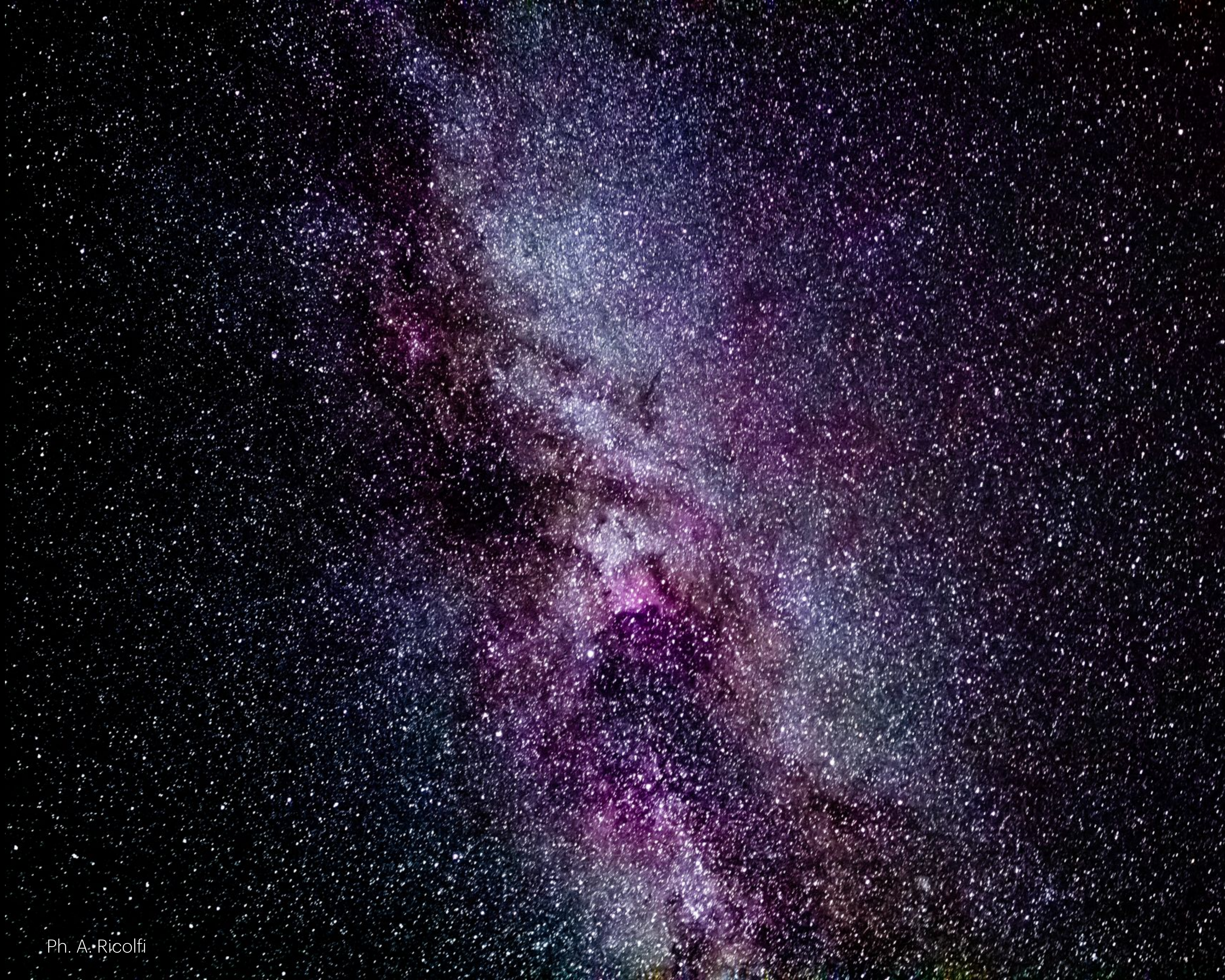
Plan

1. Motivation

2. An overview of resurgence

3. Application to complex Chern-Simons on the complement of a hyperbolic knot

4. Conclusions



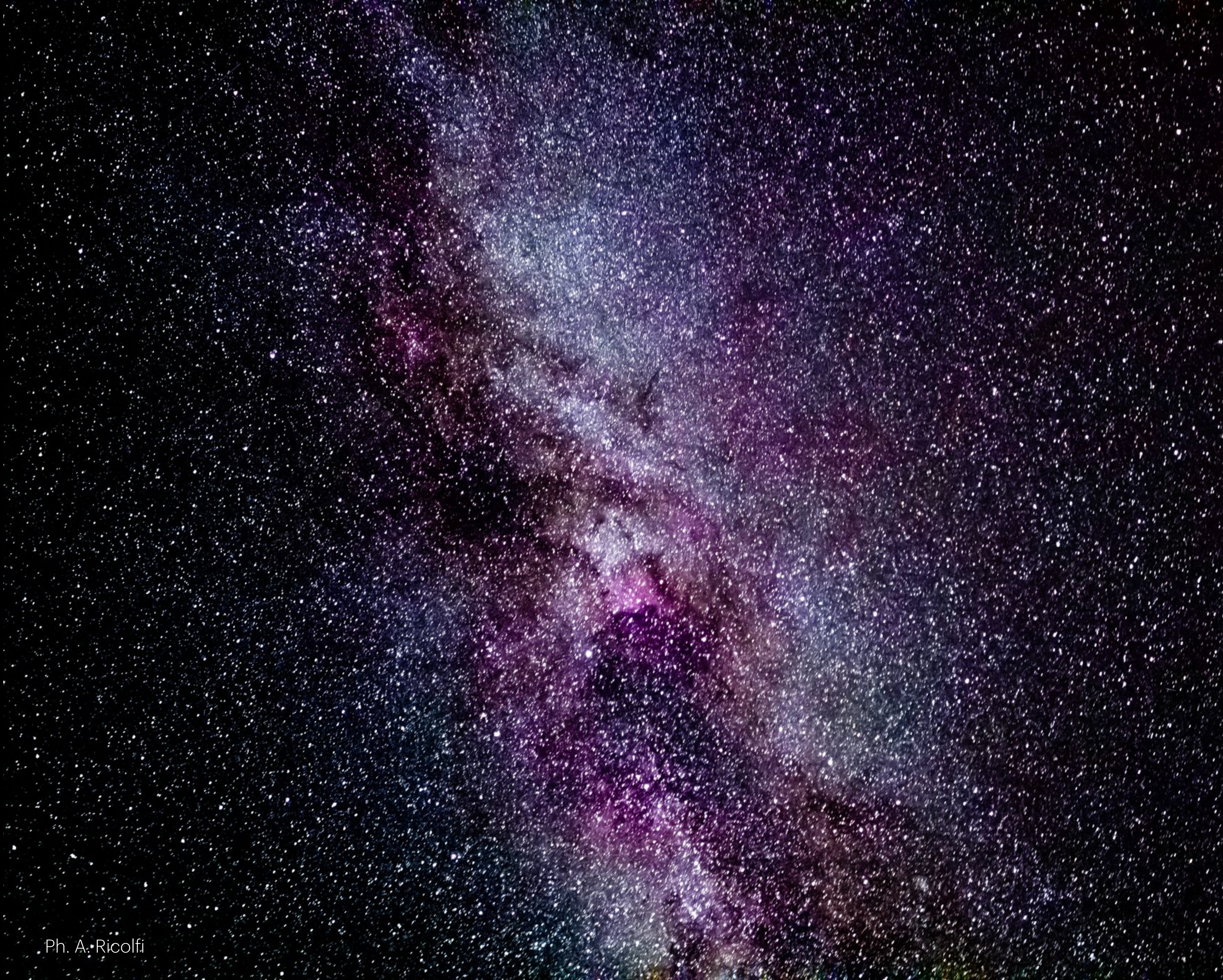
Plan

1. Motivation

2. An overview of resurgence

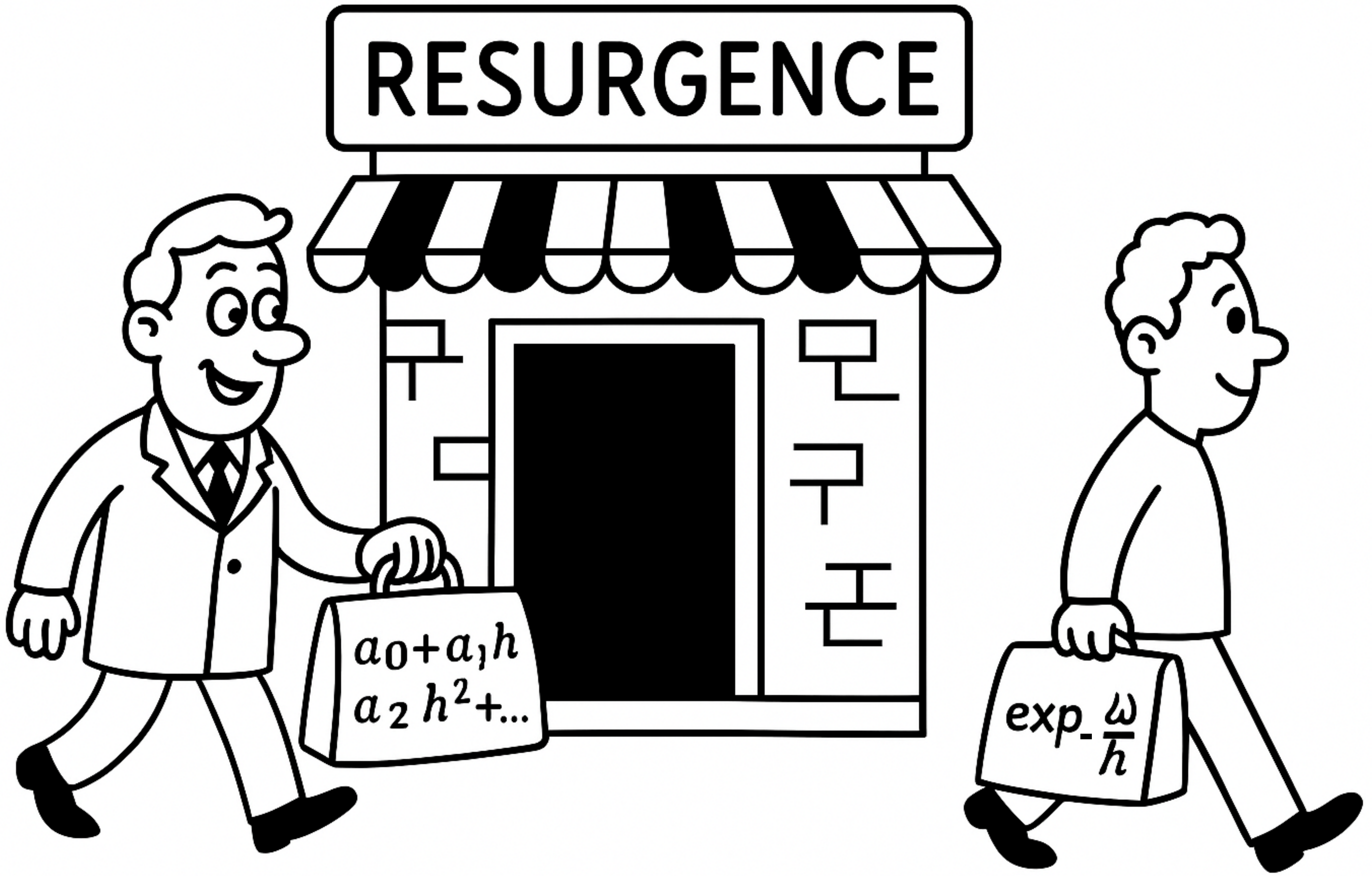
3. Application to complex Chern-Simons on the complement of a hyperbolic knot

4. Conclusions



Ph. A. Ricolfi

Resurgence: going beyon perturbation theory



Divergent series are ubiquitous in physics

QFT

- Exact WKB
- Integrable field theories

Topological String

- Topological String/Spectral Theory (TS/ST) correspondence for toric Calabi-Yau (CY) 3-folds
- Holomorphic anomaly equation (HAE)

Complex Chern-Simons

- Quantum invariants of knots and 3-manifolds
- Dimofte-Garoufalidis perturbative invariants of hyperbolic knots

2D (Super) Gravity and JT Gravity

- Painlevé I and II
- Topological Recursion

- M. Serone, *Resurgence in integrable field theories*
- M. Mariño, *Les Houches lectures on non-perturbative topological strings*
- I. Aniceto, G. Basar, and R. Schiappa, *A Primer on Resurgent Transseries and Their Asymptotics*

Divergent series are ubiquitous in physics

Complex Chern-Simons

- Quantum invariants of knots and 3-manifolds
- Dimofte-Garoufalidis perturbative invariants of hyperbolic knots

- *M. Serone, Resurgence in integrable field theories*
- *M. Mariño, Les Houches lectures on non-perturbative topological strings*
- *I. Aniceto, G. Basar, and R. Schiappa, A Primer on Resurgent Transseries and Their Asymptotics*

Recent Applications

Resurgence and **renormalons** in the one-dimensional Hubbard model

Resurgence and **renormalons** in **integrable sigma models**

On the Structure of **Trans-Series in Quantum Field Theory**

Mariño, Serone, Schwick, Miravitllas, Reis, Sberveglieri, Di Pietro, ...

Resurgent structure of Topological String on **toric Calabi-Yau** threefolds

Resurgent structure of the **Refined Topological String** on **non-compact** threefolds

Resurgent structure of **Topological String** on **compact Calabi-Yau** threefolds

Gu, Mariño, Grassi, Pioline, Alexandrov, Kashani—Poor, Klemm, Aniceto, Alim, Saha, Teschner, Tulli, Rella, **VF**, ...

Resurgent structure of complex Chern-Simons on the complement of **hyperbolic knots**

Resurgent structure of complex Chern-Simons on **Siefert fibered spheres**

Garoufalidis, Mariño, Gu, Wheeler, Andersen, Mistergard, Sauzin, **VF**, ...

Resurgence in 2D **Quantum Gravity** and 2D **Quantum Super Gravity** (SUGRA)

Resurgence in JT Gravity (*in progress*)

Schiappa, Eynard, Iwaki, Mariño, Schwick, Bridgeland, ...

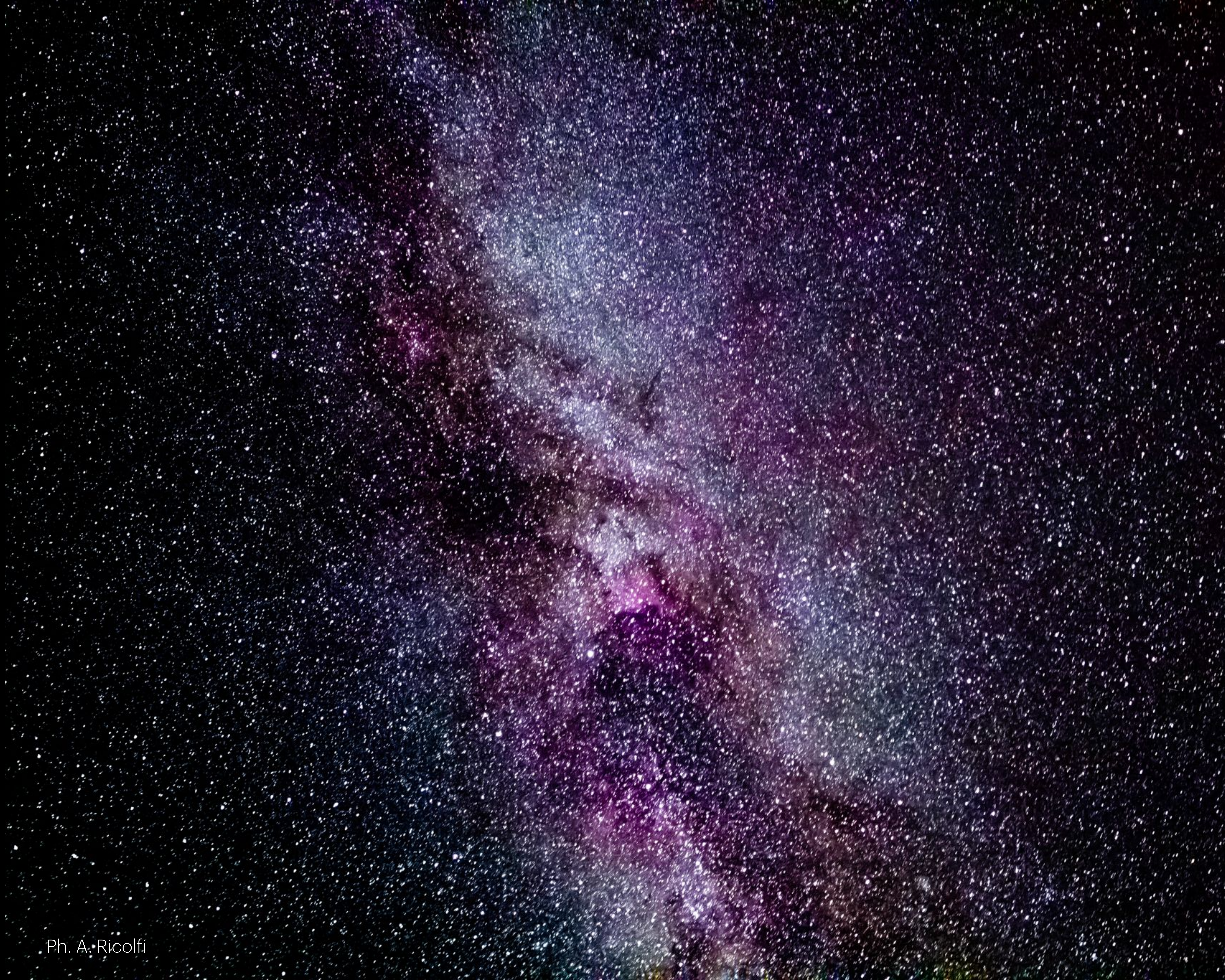
Plan

1. Motivation

2. An overview of resurgence

3. Application to complex Chern-Simons on the complement of a hyperbolic knot

4. Conclusions



Ph. A. Ricolfi

The Borel transform

Definition: The **Borel transform** is a formal map $\mathscr{B}: \mathbb{C}[\![\hbar]\!] \to \mathbb{C}\delta + \mathbb{C}[\![\zeta]\!]$

$$\mathscr{B}\left[\hbar^k\right] := \frac{\zeta^{k-1}}{\Gamma(k+1)}, \qquad k \in \mathbb{Q}\backslash\{0,-1,-2,\ldots\}$$

and then extend by countable linearity. Furthermore, one defines $\mathscr{B}[1] := \delta$ and δ is the convolution unit

The Borel transform is the formal inverse of the Laplace transform

$$\mathscr{L}\left[\zeta^{k-1}\right] := \int_0^\infty e^{-\zeta/\hbar}\,\zeta^{k-1} d\zeta = \Gamma(k+1)\,\hbar^k$$

In particular, we can deduce that for some $\omega \in \mathbb{C}$

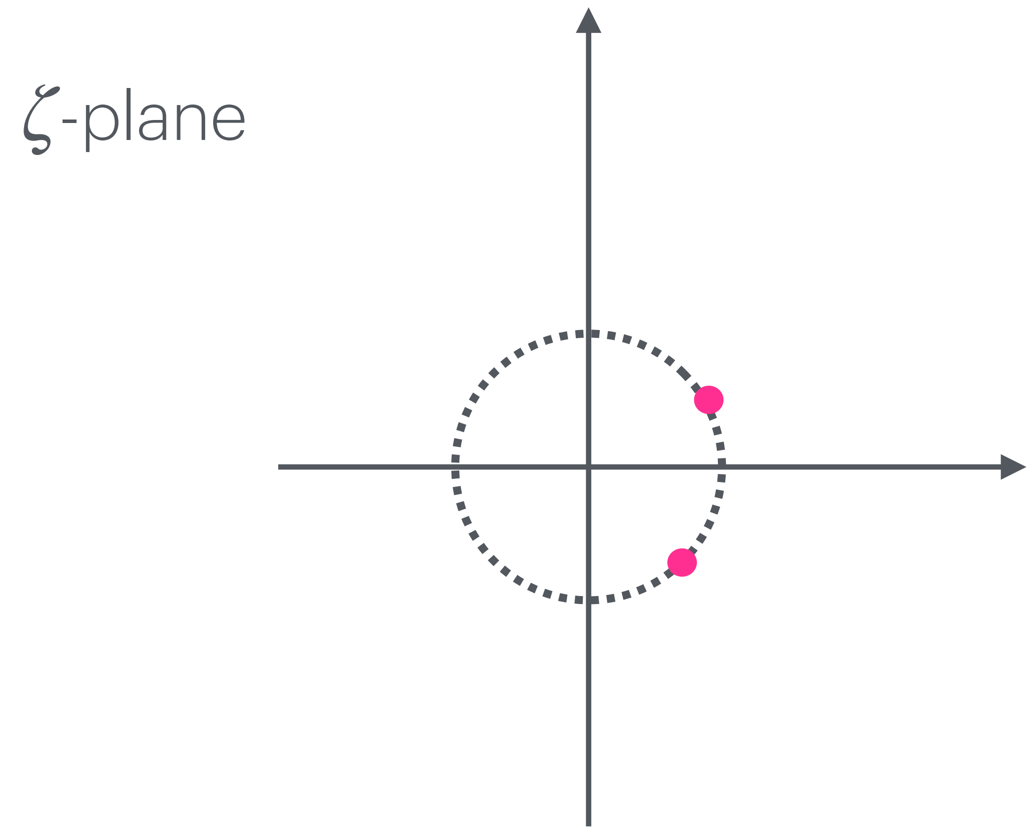
$$\mathscr{B}\left[\hbar^k e^{-\frac{\omega}{\hbar}}\right] = \frac{\zeta_\omega^{k-1}}{\Gamma(k+1)}, \qquad k \in \mathbb{Q}\backslash\{0,-1,-2,\ldots\}, \quad \zeta_\omega = \zeta - \omega$$

Gevrey series

Definition: A formal series $\tilde{\Phi}(\hbar) = \sum_{n=0}^{\infty} a_n \hbar^n \in \mathbb{C}[[\hbar]]$ is **Gevrey-1** if its coefficients a_n grow as

$$|a_n| \leq C A^n n!\,, \quad C, A > 0$$

A series $\tilde{\Phi} \in \mathbb{C}[[\hbar]]$ is Gevrey-1 if and only if its Borel transform $\tilde{\phi} := \mathscr{B}\tilde{\Phi} \in \mathbb{C}\{\zeta\}$ has a finite radius of convergence

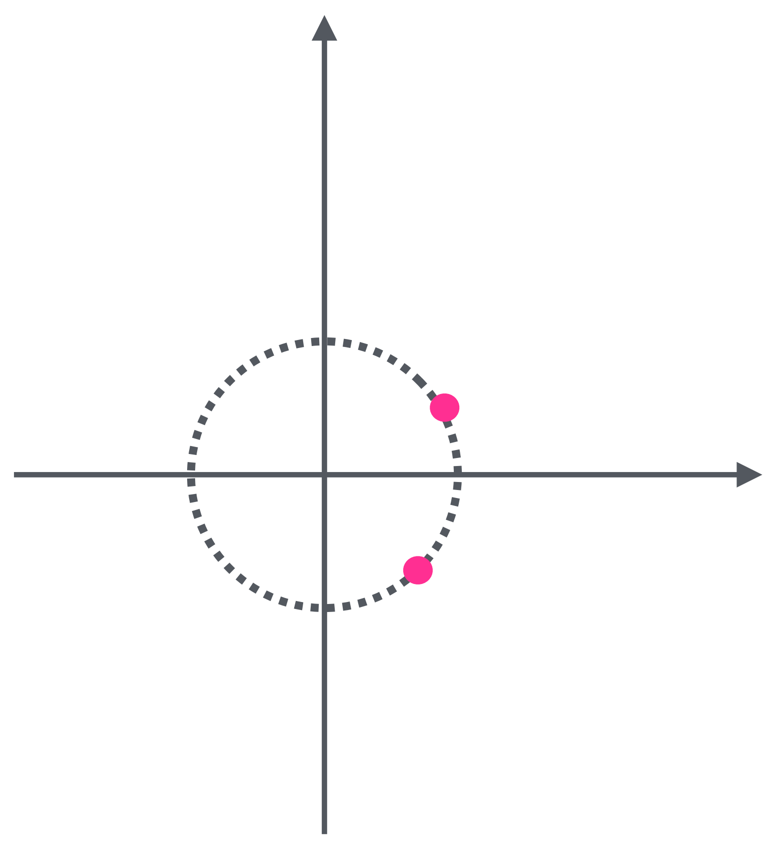


Resurgent series

Definition [Écalle]: A Gevrey-1 series $\tilde{\Phi}(\hbar) \in \mathbb{C}[[\hbar]]$ is resurgent if its Borel transform $\tilde{\phi}(\zeta) \in \mathbb{C}\{\zeta\}$ has ***endless analytic continuation***

For every $L > 0$, there exists a discrete subset $\Omega_L \subset \mathbb{C}$ such that $\tilde{\phi}(\zeta)$ can be analytically continued along every path of length less than L, which avoids Ω_L starting from the same point ζ_0

The set of singularities is $\Omega = \bigcup_L \Omega_L$

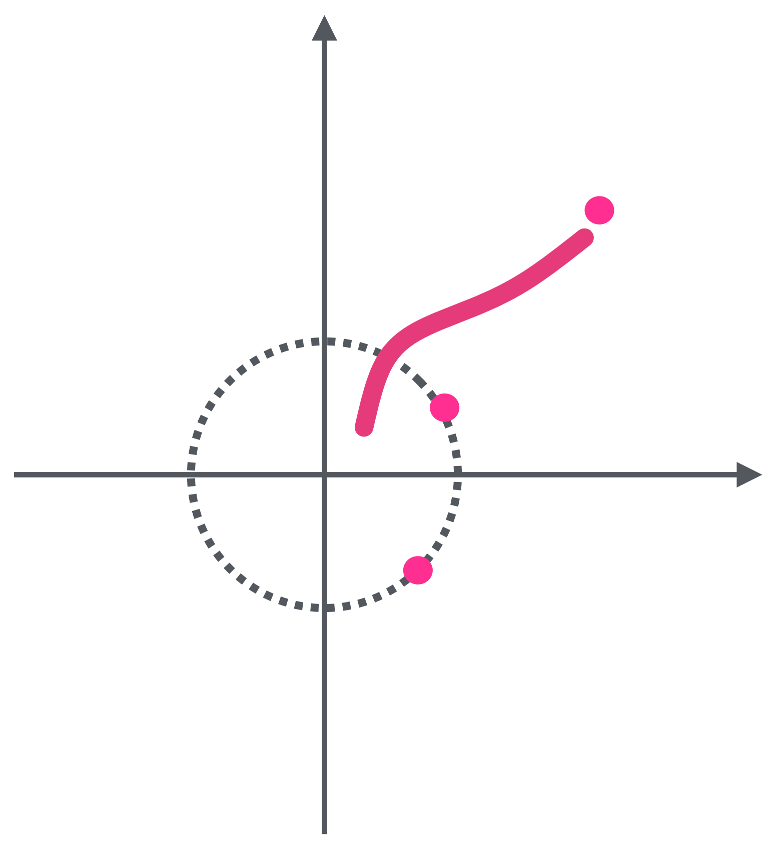


Resurgent series

Definition [Écalle]: A Gevrey-1 series $\tilde{\Phi}(\hbar) \in \mathbb{C}[[\hbar]]$ is resurgent if its Borel transform $\tilde{\phi}(\zeta) \in \mathbb{C}\{\zeta\}$ has ***endless analytic continuation***

For every $L > 0$, there exists a discrete subset $\Omega_L \subset \mathbb{C}$ such that $\tilde{\phi}(\zeta)$ can be analytically continued along every path of length less than L, which avoids Ω_L starting from the same point ζ_0

The set of singularities is $\Omega = \bigcup_L \Omega_L$

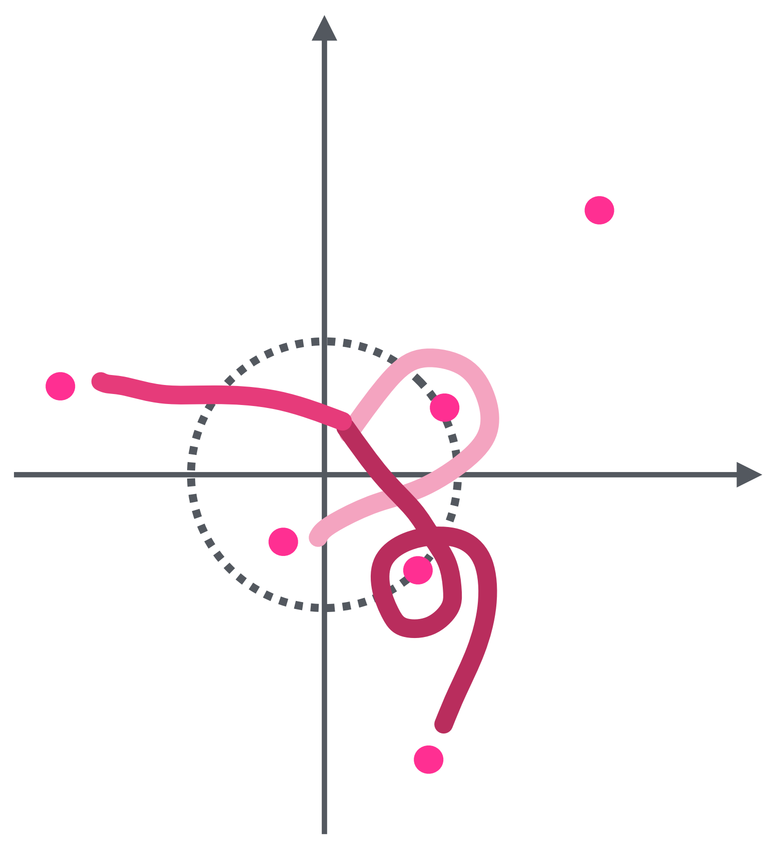


Resurgent series

Definition [Écalle]: A Gevrey-1 series $\tilde{\Phi}(\hbar) \in \mathbb{C}[[\hbar]]$ is resurgent if its Borel transform $\tilde{\phi}(\zeta) \in \mathbb{C}\{\zeta\}$ has ***endless analytic continuation***

For every $L > 0$, there exists a discrete subset $\Omega_L \subset \mathbb{C}$ such that $\tilde{\phi}(\zeta)$ can be analytically continued along every path of length less than L, which avoids Ω_L starting from the same point ζ_0

The set of singularities is $\Omega = \bigcup_L \Omega_L$



New information hidden at the singularities

Example: the Euler series

$$\tilde{\Phi}(\hbar) = \sum_{n=0}^{\infty} n!\,\hbar^{n+1}$$

$$\Big\downarrow \mathscr{B}$$

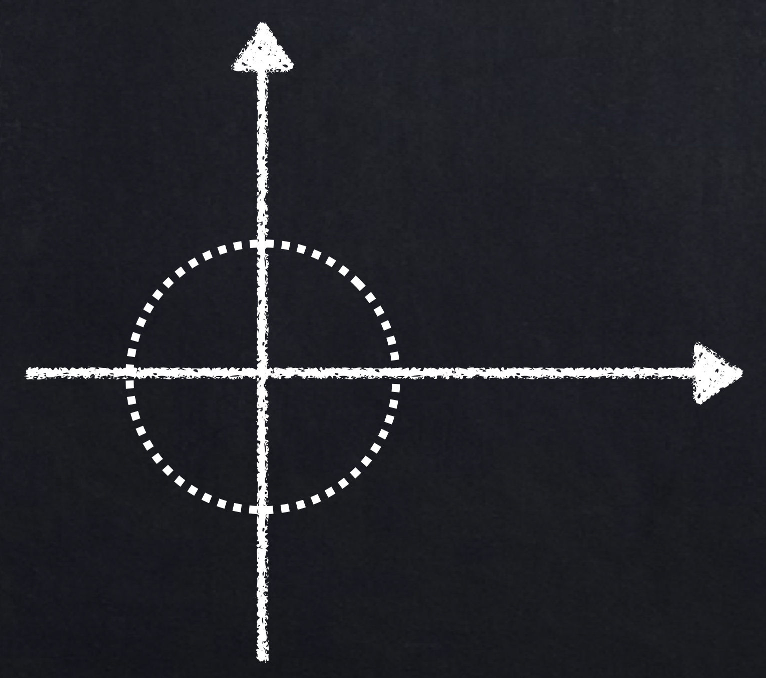
$$\tilde{\phi}(\zeta) = \sum_{n=0}^{\infty} n!\frac{\zeta^n}{n!}$$

Example: the Euler series

$$\tilde{\Phi}(\hbar) = \sum_{n=0}^{\infty} n!\,\hbar^{n+1}$$

$$\big\downarrow \mathscr{B}$$

$$\tilde{\phi}(\zeta) = \sum_{n=0}^{\infty} \zeta^n \in \mathbb{C}\{\zeta\}$$

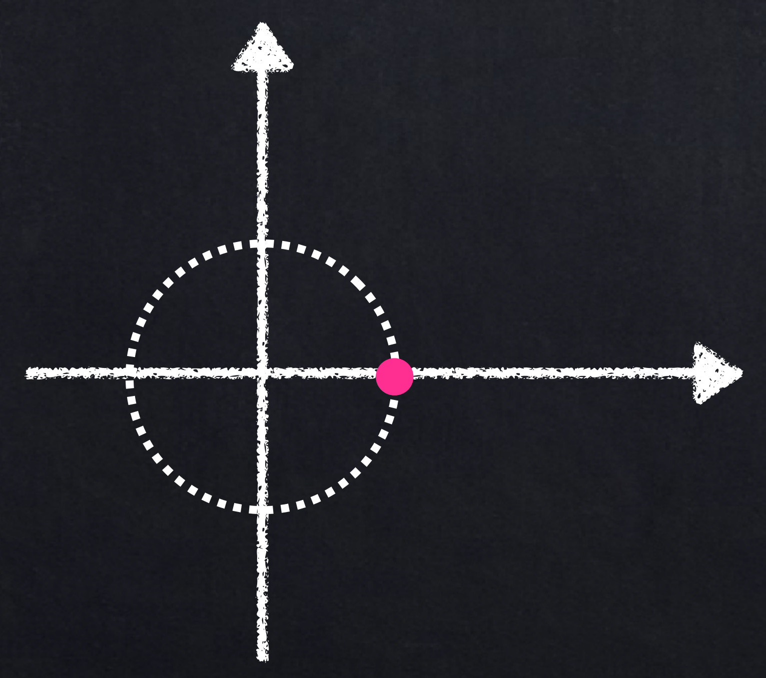


Example: the Euler series

$$\tilde{\Phi}(\hbar) = \sum_{n=0}^{\infty} n!\,\hbar^{n+1}$$

$$\Big\downarrow \mathscr{B}$$

$$\tilde{\phi}(\zeta) = \sum_{n=0}^{\infty} \zeta^n \in \mathbb{C}\{\zeta\} \xrightarrow[\text{sum}]{} \hat{\phi}(\zeta) = \frac{1}{1-\zeta}$$



simple pole at $\zeta = 1$

(Counter)Example: Gevrey-1 but not resurgent

$$\tilde{\Phi}(\hbar) = \sum_{n=0}^{\infty} a_n \, n! \hbar^{n+1} \qquad\qquad a_n = \begin{cases} 1 & n = 2^k \\ 0 & \end{cases}$$

$$\begin{tikzcd} \tilde{\Phi}(\hbar) \arrow[d, "\mathscr{B}"'] \\ \tilde{\phi}(\zeta) \end{tikzcd}$$

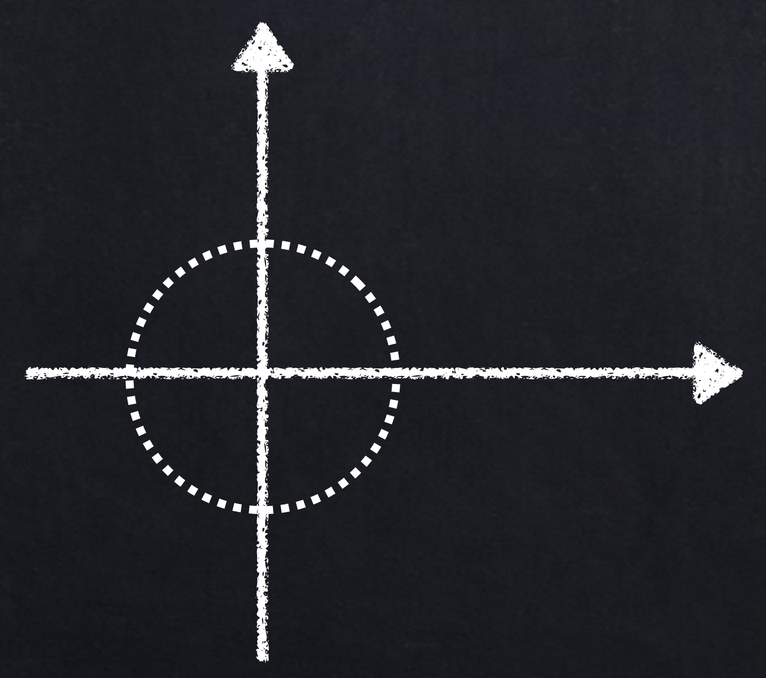
$$\tilde{\phi}(\zeta) = \sum_{n=0}^{\infty} a_n \, n! \frac{\zeta^n}{n!}$$

(Counter)Example: Gevrey-1 but not resurgent

$$\tilde{\Phi}(\hbar) = \sum_{n=0}^{\infty} a_n \, n! \hbar^{n+1} \qquad\qquad a_n = \begin{cases} 1 & n = 2^k \\ 0 & \end{cases}$$

$$\Big\downarrow \mathcal{B}$$

$$\tilde{\phi}(\zeta) = \sum_{n=0}^{\infty} a_n \, \zeta^n$$

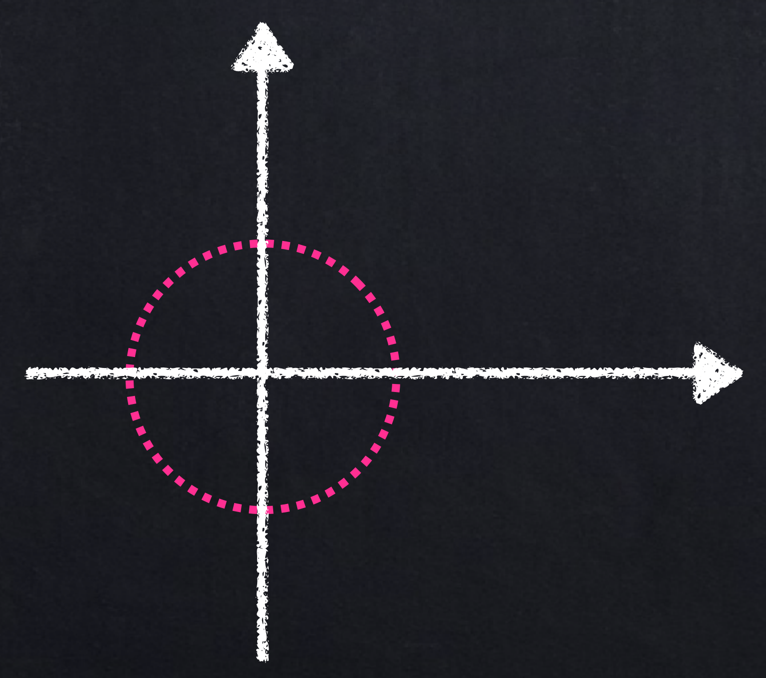


(Counter)Example: Gevrey-1 but not resurgent

$$\tilde{\Phi}(\hbar) = \sum_{n=0}^{\infty} a_n \, n! \hbar^{n+1} \qquad a_n = \begin{cases} 1 & n = 2^k \\ 0 & \end{cases}$$

$$\Big\downarrow \mathcal{B}$$

$$\tilde{\phi}(\zeta) = \sum_{n=0}^{\infty} a_n \, \zeta^n$$



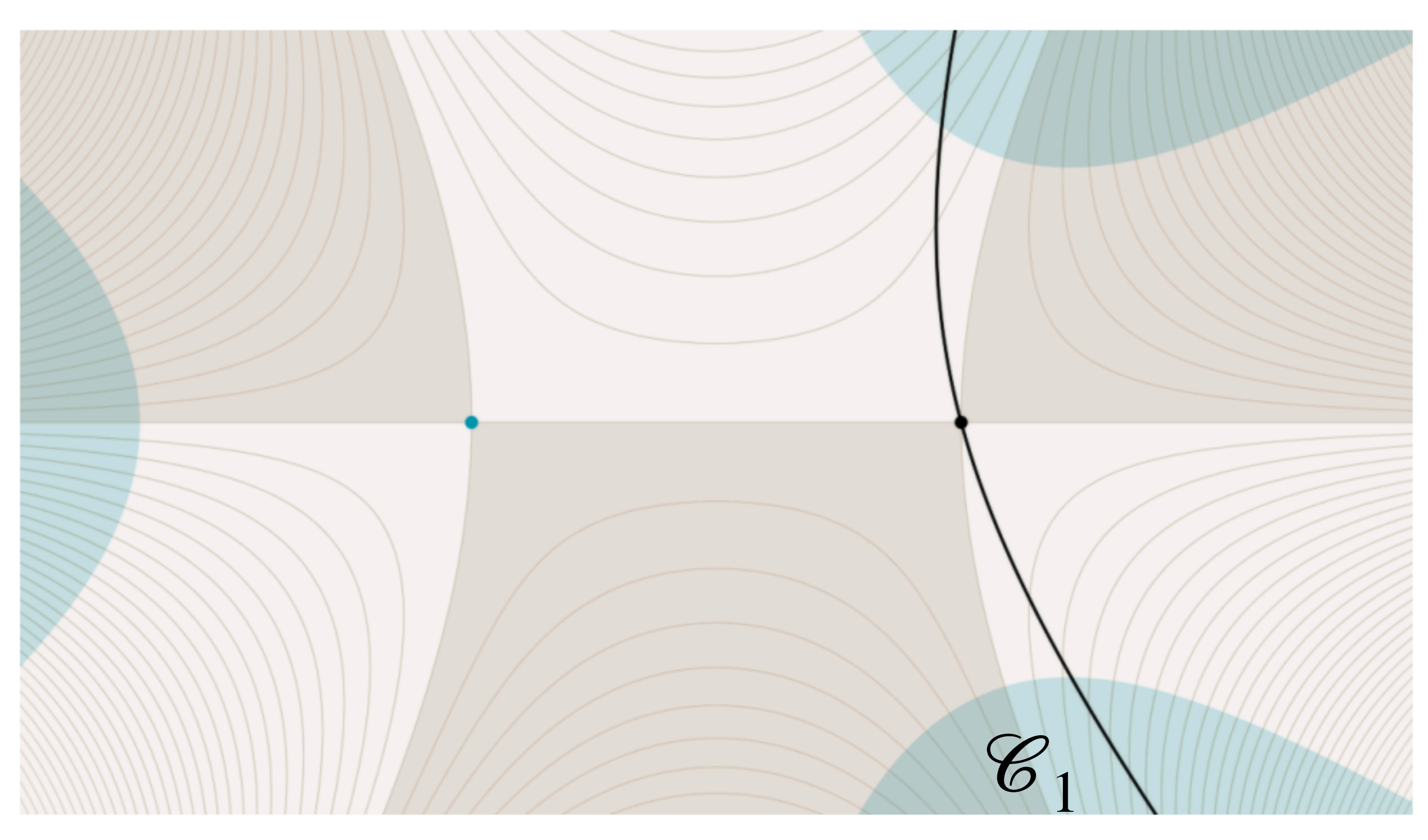
$\zeta_k^\ell = e^{\frac{2\pi i \ell}{2^k}}$ is a singular point for every $\ell \in \mathbb{Z}, k \in \mathbb{Z}_{>0}$

$\Rightarrow \tilde{\phi}$ can't be analytically continued

Example: the Airy function

The Airy function is a solution of a linear 2nd order differential equation

$$\mathrm{Ai}(\hbar) = \int_{\mathscr{C}_1} e^{-\frac{4t^3-3t}{\hbar}}\,dt \;\sim_{\hbar\to 0}\; \tilde{\Phi}_1(\hbar) \;=\; e^{-1/\hbar}\,\hbar^{1/2}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{6}\right)_k\left(\frac{5}{6}\right)_k}{2^k k!}(-\hbar)^k, \quad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$$

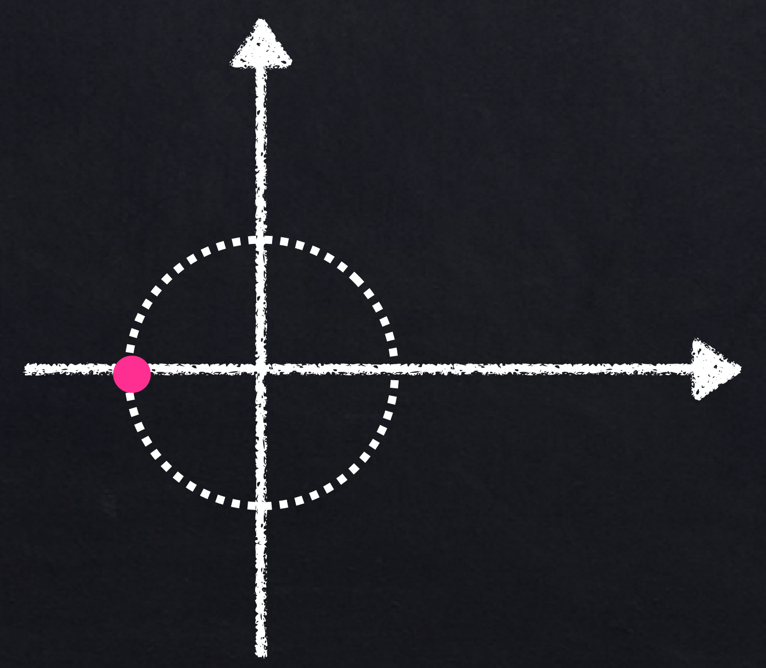


Example: the Airy function

$$\tilde{\Phi}_1(\hbar) \;=\; e^{-1/\hbar}\,\hbar^{1/2} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{6}\right)_k \left(\frac{5}{6}\right)_k}{2^k k!} (-\hbar)^k$$

$$\Big\downarrow \mathscr{B}$$

$$\tilde{\phi}_1(\zeta) \;=\; \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{6}\right)_k \left(\frac{5}{6}\right)_k}{2^k k!} \frac{\zeta_1^{k-\frac{1}{2}}}{\Gamma\left(k+\frac{1}{2}\right)} \quad \xrightarrow{\text{sum}} \quad \hat{\phi}_1(\zeta) \;=\; \zeta_1^{-1/2}\, {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; -\frac{\zeta_1}{2}\right)$$



Logarithmic singularity at $\zeta = -1$

Example: the Airy function

Taking the analytic continuation at the singular point $\zeta = -1$

$$\hat{\phi}_1(\zeta + \mathrm{i}\epsilon) - \hat{\phi}_1(\zeta - \mathrm{i}\epsilon) \ = S_1 \ \zeta_{-1}^{-1/2} \ {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \frac{\zeta_{-1}}{2}\right)$$

$$S_1 \in \mathbb{C} \qquad\qquad \zeta_{-1}^{-1/2} \ {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \frac{\zeta_{-1}}{2}\right) = \hat{\phi}_2(\zeta)$$

where $\hat{\phi}_2$ defines the analytic continuation of a different germ $\tilde{\phi}_2(\zeta) \in \mathbb{C}\{\zeta\}$

Example: the Airy function

Taking the analytic continuation at the singular point $\zeta = -1$

$$\hat{\phi}_1(\zeta + \mathrm{i}\epsilon) - \hat{\phi}_1(\zeta - \mathrm{i}\epsilon) \ = S_1 \ \zeta_{-1}^{-1/2} \ {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \frac{\zeta_{-1}}{2}\right)$$

$$S_1 \in \mathbb{C} \qquad\qquad \zeta_{-1}^{-1/2} \ {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \frac{\zeta_{-1}}{2}\right) = \hat{\phi}_2(\zeta)$$

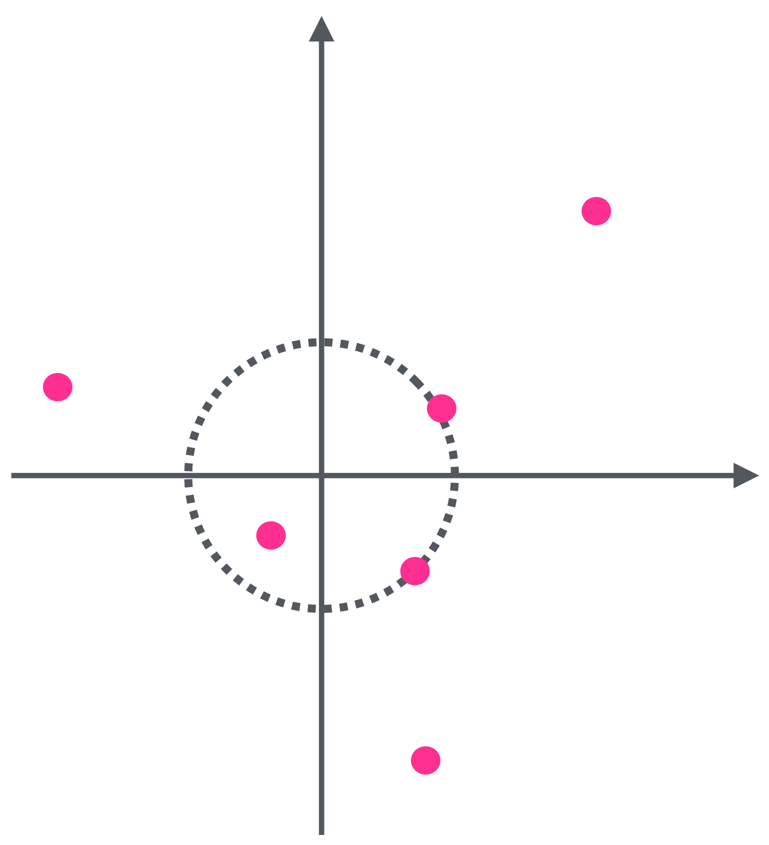
where $\hat{\phi}_2$ defines the analytic continuation of a different germ $\tilde{\phi}_2(\zeta) \in \mathbb{C}\{\zeta\}$

- Taking the analytic continuation of $\hat{\phi}_2$ at the singular point $\zeta = 1$, we recover the function $\hat{\phi}_1$ up to a constant S_2
- The new germ is the Borel transform of an independent solution of the differential equation $\tilde{\phi}_2(\zeta) = \mathscr{B}\tilde{\Phi}_2$

Resurgent structure

Let $\tilde{\Phi} \in \mathbb{C}[[\hbar]]$ be a Gevrey-1 resurgent series. The **resurgent structure** of $\tilde{\Phi}$ consists of the following data:

1. $\omega \in \Omega \subset \mathbb{C}$ the **singularities** of the Borel transform $\hat{\phi}(\zeta) \in \mathbb{C}\{\zeta\}$ in the Borel plane
2. $S_\omega \in \mathbb{C}$ the **Stokes constants** associated with each $\omega \in \Omega$
3. $\tilde{\phi}_\omega(\zeta) \in \mathbb{C}\{\zeta\}$ **new germs of analytic functions**, which resurges at each $\omega \in \Omega$

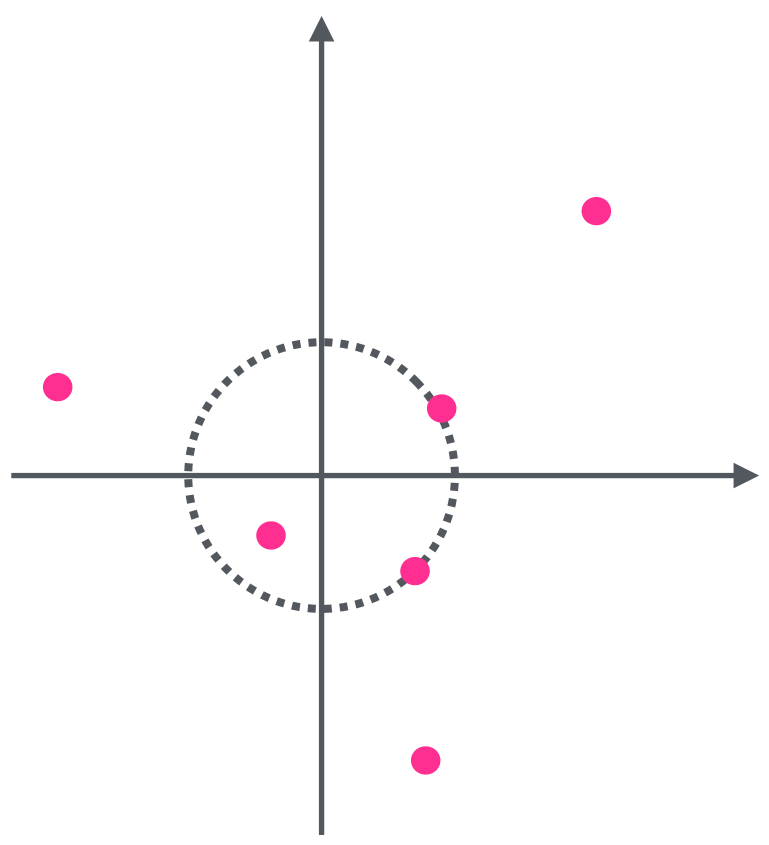


Resurgent structure

Let $\tilde{\Phi} \in \mathbb{C}[[\hbar]]$ be a Gevrey-1 resurgent series. The **resurgent structure** of $\tilde{\Phi}$ consists of the following data:

1. $\omega \in \Omega \subset \mathbb{C}$ the **singularities** of the Borel transform $\hat{\phi}(\zeta) \in \mathbb{C}\{\zeta\}$ in the Borel plane
2. $S_\omega \in \mathbb{C}$ the **Stokes constants** associated with each $\omega \in \Omega$
3. $\tilde{\phi}_\omega(\zeta) \in \mathbb{C}\{\zeta\}$ **new germs of analytic functions**, which resurges at each $\omega \in \Omega$

New series resurge at the singularities



$$\hat{\phi}(\zeta) = -\frac{S_\omega}{2\pi i}\log(\zeta-\omega)\tilde{\phi}_\omega(\zeta-\omega) + \text{ reg. } \quad \text{for } \zeta \text{ near } \omega$$

$$\hat{\phi}(\zeta) = -\frac{1}{2\pi \mathrm{i}}\frac{S_\omega}{\zeta-\omega} + \text{ reg. } \quad \text{for } \zeta \text{ near } \omega$$

Trans-series: the building blocks of non-perturbative corrections

From a simple reusrgent series to exponentially sub-leading order corrections

$$\tilde{\Phi} \in \mathbb{C}[[\hbar]] \quad \rightsquigarrow \quad e^{-\omega/\hbar} S_\omega \tilde{\Phi}_\omega(\hbar) \ \in \ e^{-\omega/\hbar}\mathbb{C}[[\hbar]]$$

Trans-series are the building blocks of non-perturbative corrections, defined by a sequence of sub-leading order corrections

$$Z(\hbar;\sigma) \ = \sum_{k\geq 0} \sigma^k \, e^{-k\,\omega/\hbar} \, \tilde{\Phi}_\omega(\hbar)$$

where $\sigma \in \mathbb{C}$ is a free parameter

The **Stokes automorphisms** act as automorphisms on the space of trans-series, and compute the Stokes constants

Solutions of non-linear ODEs are usually trans-series [Baldino-Schiappa-Schwick-Vega, Mariño-Miravitllas, Delabaere]

Resurgence in practice

When it comes to doing explicit computations, we usually face the following situations:

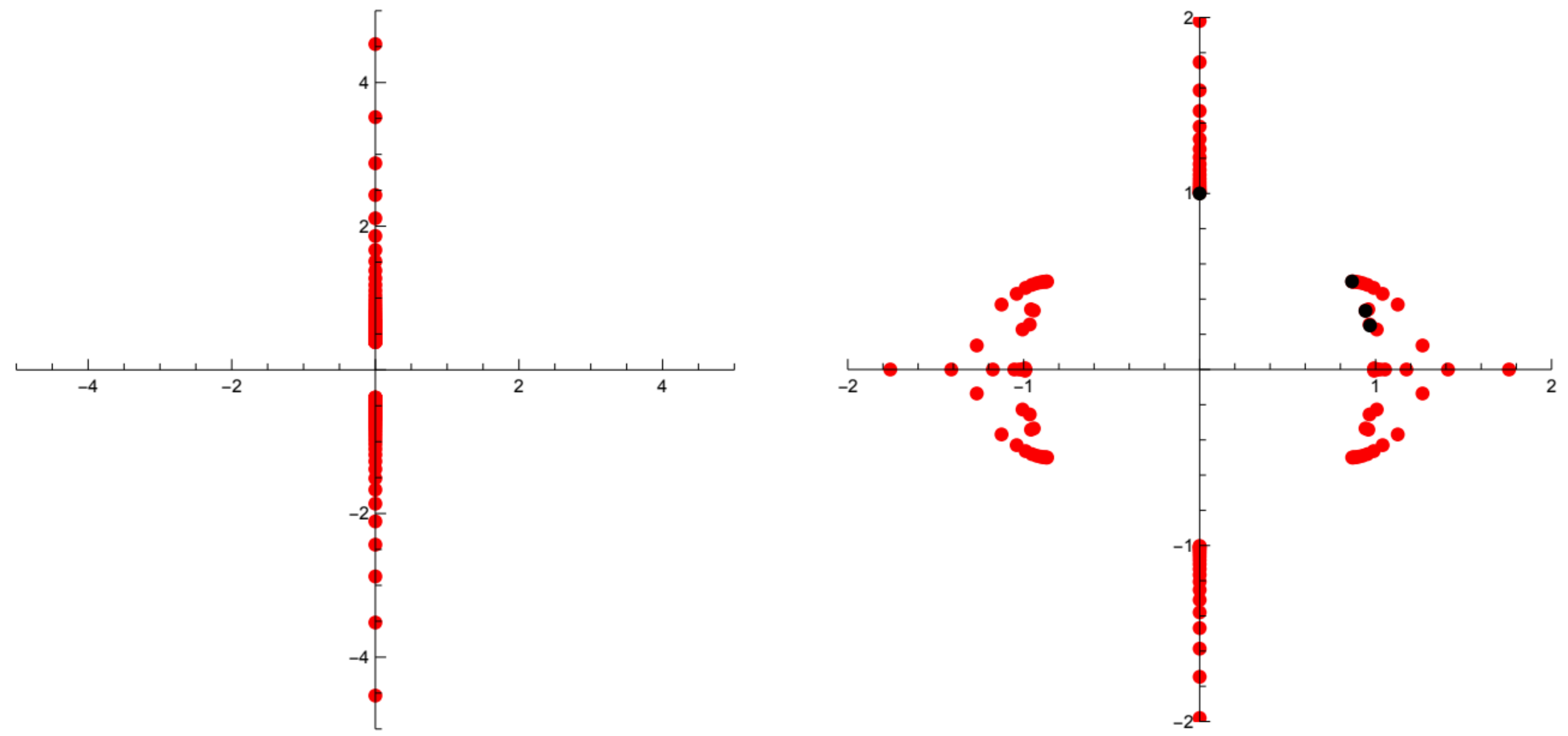
- If the perturbative coefficients are known in closed form, then one can use the **Hadamard product**

- If many ($N \geq 100$) perturbative coefficients are available with high precision, then one can use **Borel-Padé**

$$\tilde{P}(\hbar) = \sum_{n=1}^{N} a_n \hbar^n \xrightarrow{\mathscr{B}} \tilde{p}(\zeta) = \sum_{n=1}^{N} \frac{a_n}{n!} \zeta^n \xrightarrow{\text{Padé}} \hat{p}(\zeta) = \frac{q_1(\zeta)}{q_2(\zeta)},$$

where $q_1, q_2 \in \mathbb{C}[\zeta]$

Plotting the zeros of $q_2(\zeta)$ we see how singularities in the ζ-plane distribute



Gu-Mariño, ArXiv: 2211.01403

Summability

Summation methods associate to a divergent series $\tilde{\Phi}$ an analytic function $\hat{\Phi}$

$$\tilde{\Phi}(\hbar) \in \mathbb{C}[[\hbar]] \xrightarrow{???} \hat{\Phi} \text{ analytic}$$

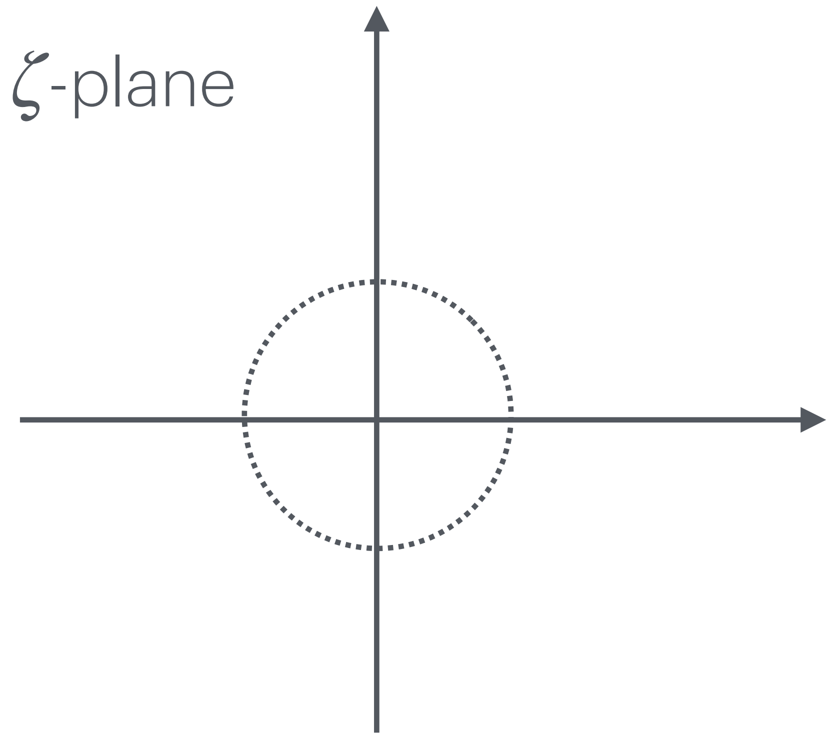
From formal to analytic

Borel-Laplace summation

Borel-Laplace summation works in three steps

1. The formal **Borel transform** $\mathscr{B}$

$$\begin{tikzcd} \tilde{\Phi}(\hbar) \in \mathbb{C}[[\hbar]] \arrow[r, dashed] \arrow[d, "\mathscr{B}"'] & \hat{\Phi}(\hbar) \text{ analytic} \\ \tilde{\phi}(\zeta) \in \mathbb{C}\{\zeta\} & \end{tikzcd}$$

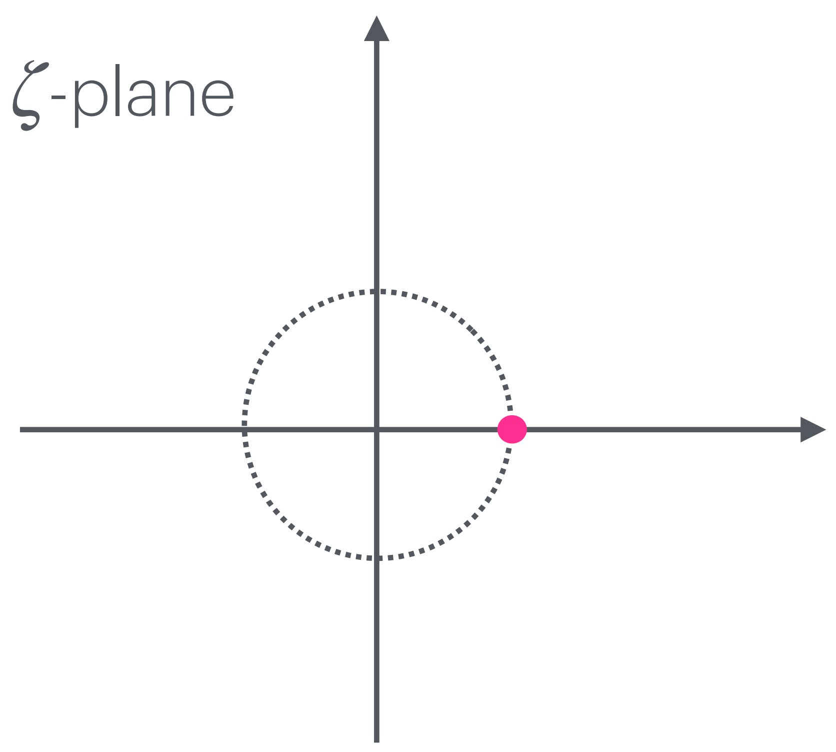


Borel-Laplace summation

Borel-Laplace summation works in three steps

2. **Analytic continuation** in the ζ-plane

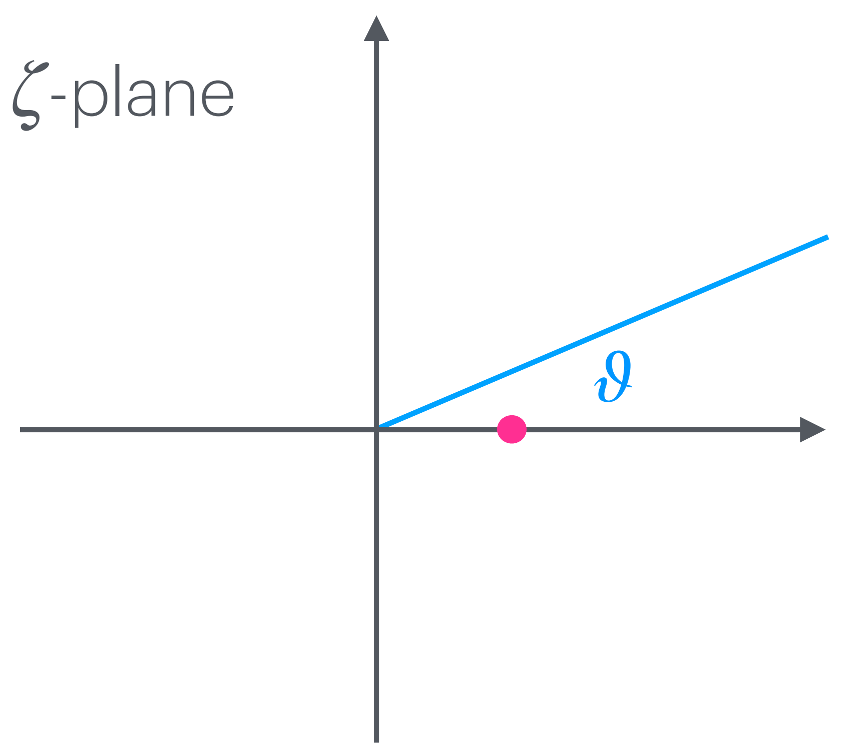
$$
\begin{tikzcd}
\tilde{\Phi}(\hbar) \in \mathbb{C}[[\hbar]] \arrow[r, dashed] \arrow[d, "\mathscr{B}"'] & \hat{\Phi}(\hbar) \text{ analytic} \\
\tilde{\phi}(\zeta) \in \mathbb{C}\{\zeta\} \arrow[r, "\text{sum}"'] & \hat{\phi}(\zeta) \text{ analytic}
\end{tikzcd}
$$



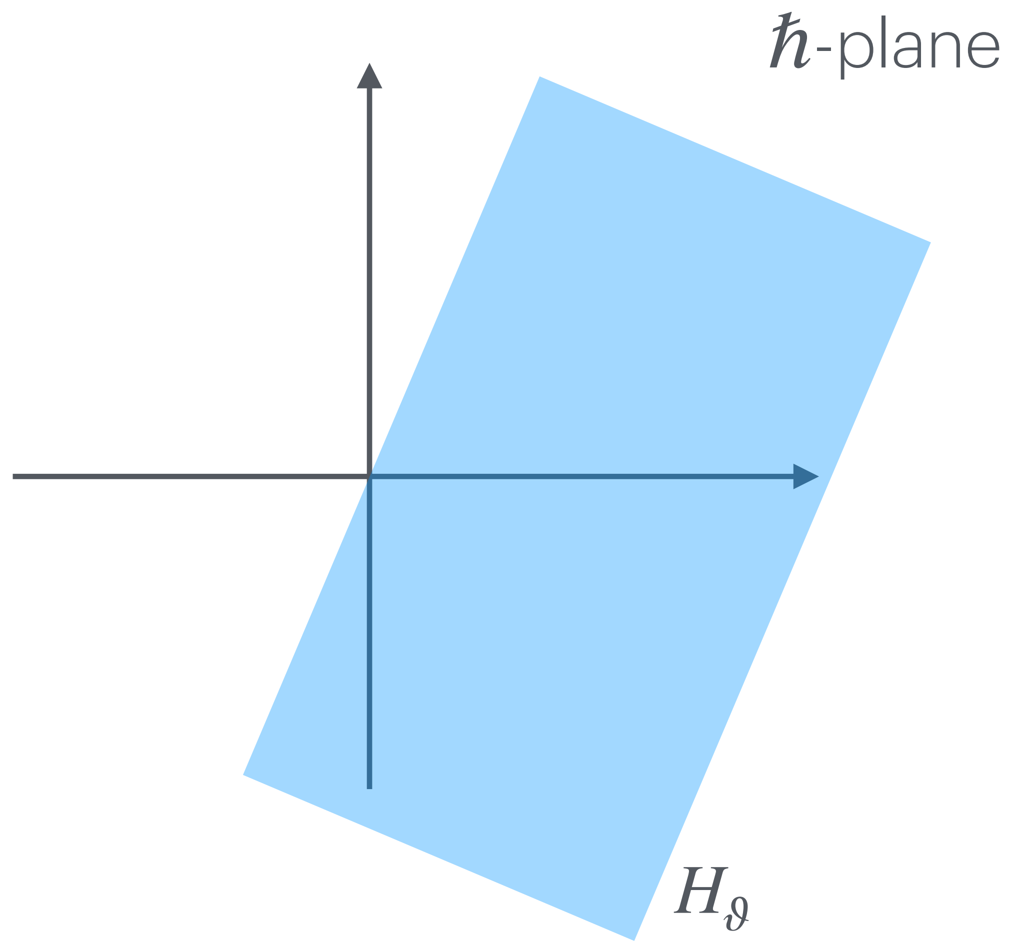
Borel-Laplace summation

Borel-Laplace summation works in three steps

3. The **Laplace transform** $\mathscr{L}^{\vartheta}$ is defined along a ray in the direction ϑ that avoids the singularities



$$
\begin{tikzcd}
\tilde{\Phi}(\hbar) \in \mathbb{C}[[\hbar]] \arrow[r, "\text{Borel-Laplace sum}"] \arrow[d, "\mathscr{B}"'] & \hat{\Phi}_{\vartheta}(\hbar) \text{ analytic} \\
\tilde{\phi}(\zeta) \in \mathbb{C}\{\zeta\} \arrow[r, "\text{sum}"'] & \hat{\phi}(\zeta) \text{ analytic} \arrow[u, "\mathscr{L}^{\vartheta}"']
\end{tikzcd}
$$

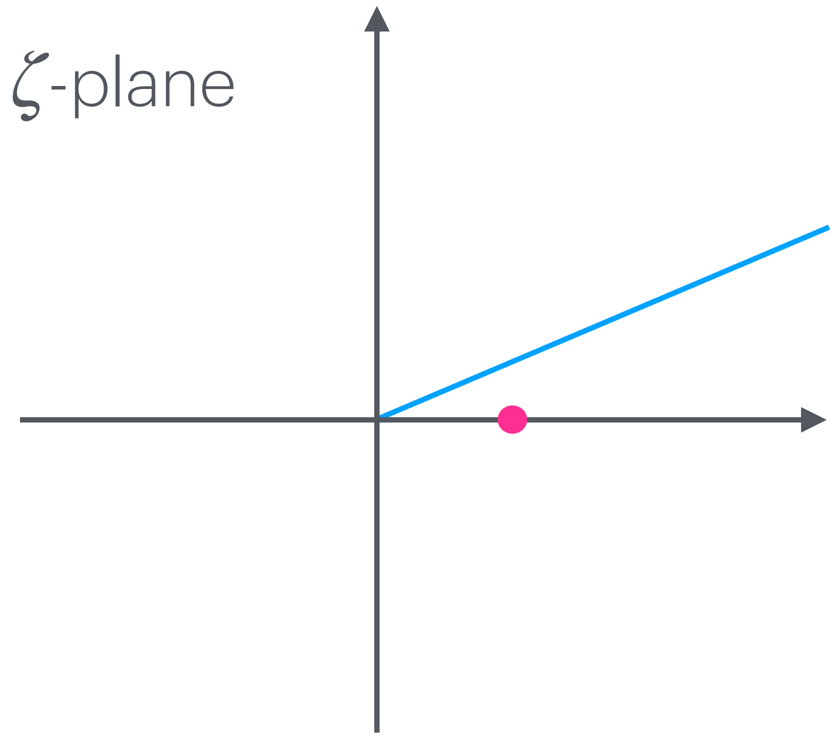


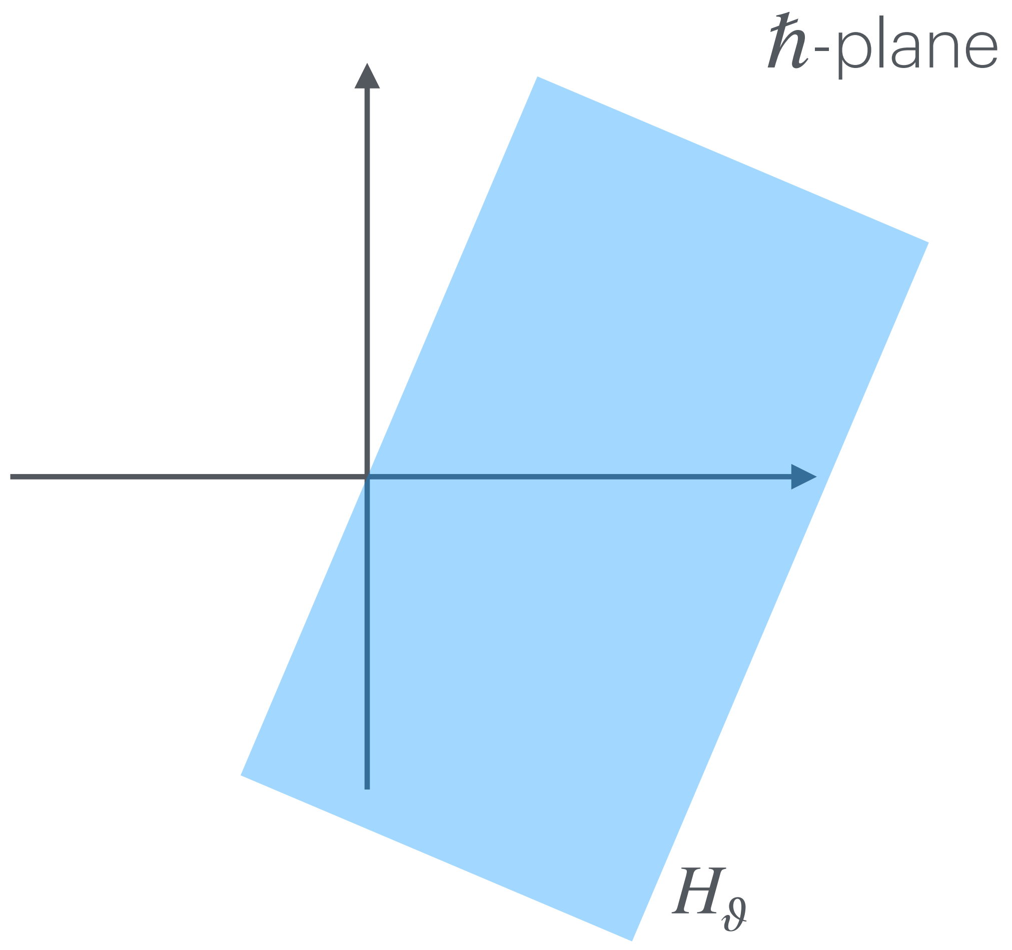
Borel-Laplace summation

Borel-Laplace summation works in three steps

The Borel-Laplace sum $\hat{\Phi}$ is uniform (Gevrey) asymptotic to $\tilde{\Phi}$

$$\begin{tikzcd}
\tilde{\Phi}(\hbar) \in \mathbb{C}[[\hbar]] \arrow[r, shift left, "\text{Borel-Laplace sum}"] \arrow[d, "\mathscr{B}"'] & \hat{\Phi}_\vartheta(\hbar) \text{ analytic} \arrow[l, shift left, "\text{asymptotics}"] \\
\tilde{\phi}(\zeta) \in \mathbb{C}\{\zeta\} \arrow[r, "\text{sum}"'] & \hat{\phi}(\zeta) \text{ analytic} \arrow[u, "\mathscr{L}^\vartheta"']
\end{tikzcd}$$

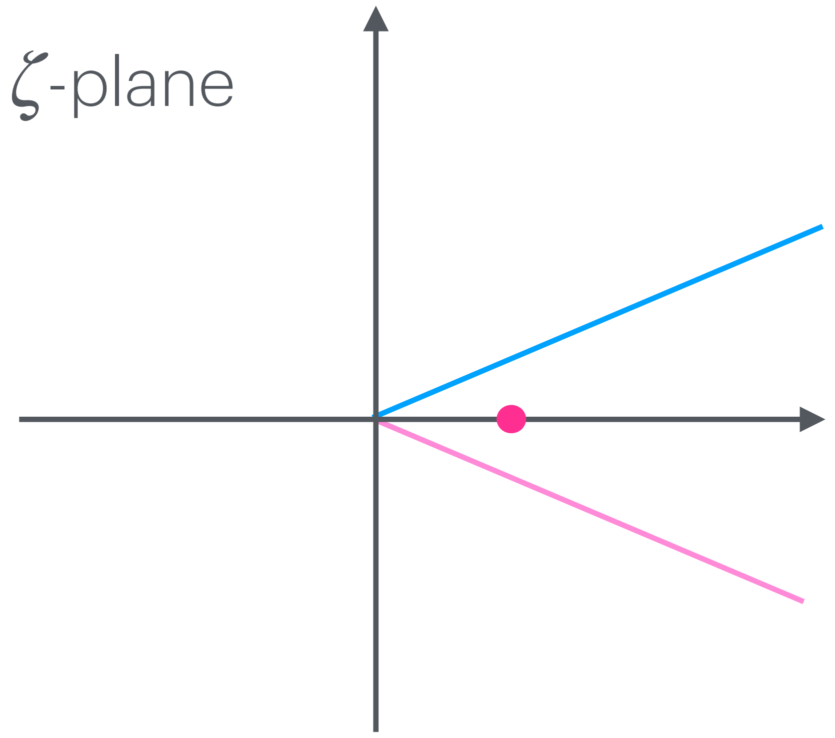


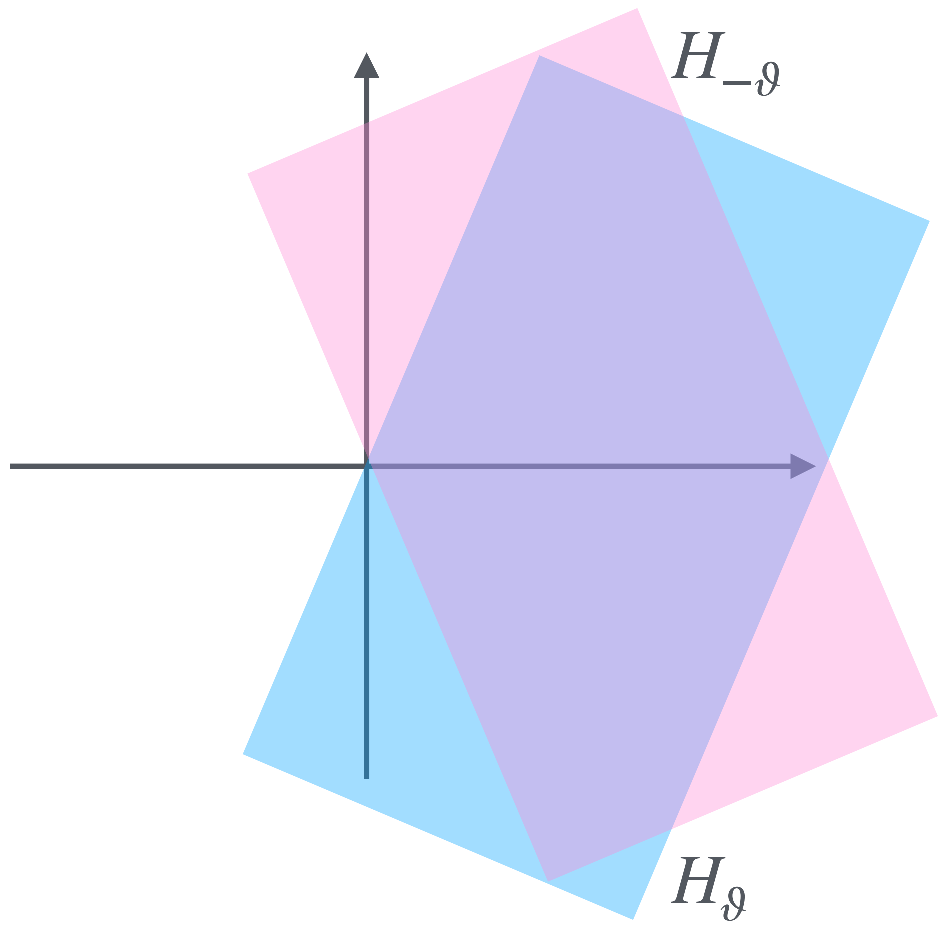


Divergence is due to exponentially suppressed terms

What is the effect of the singularities?

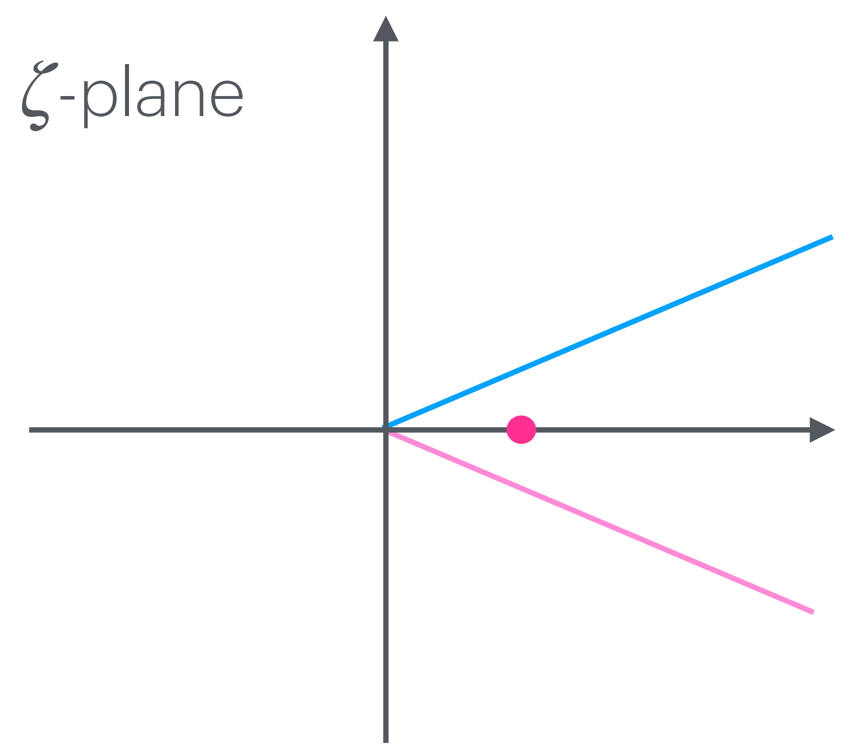
The exponentially small terms can be reconstructed from the divergent series

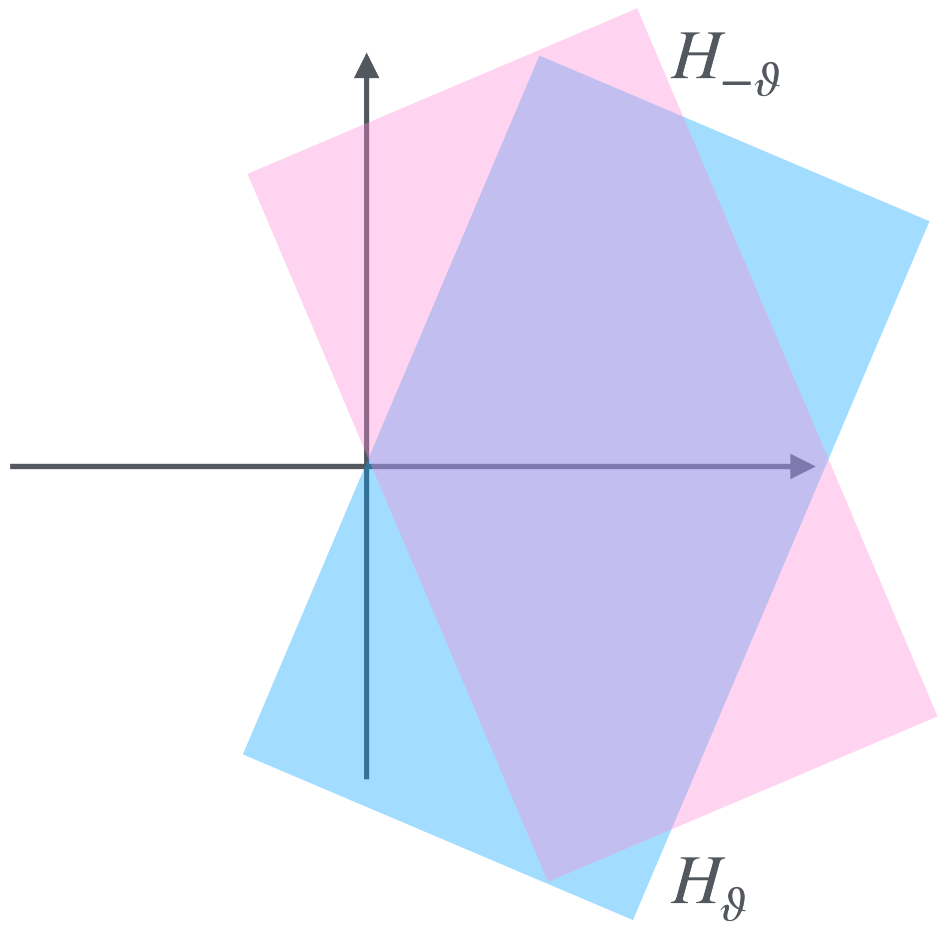




What is the effect of the singularities?

The exponentially small terms can be reconstructed from the divergent series



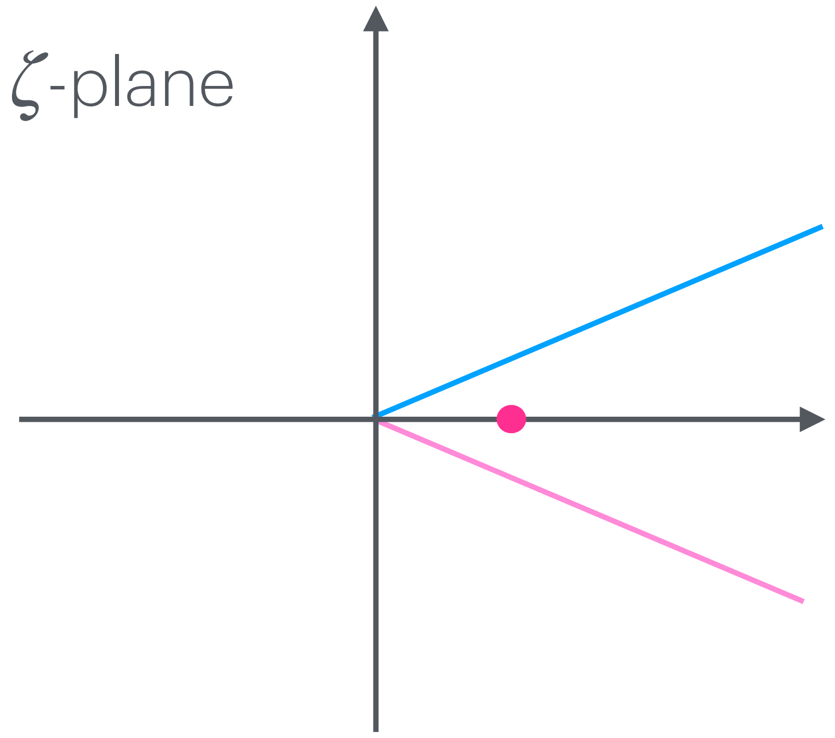


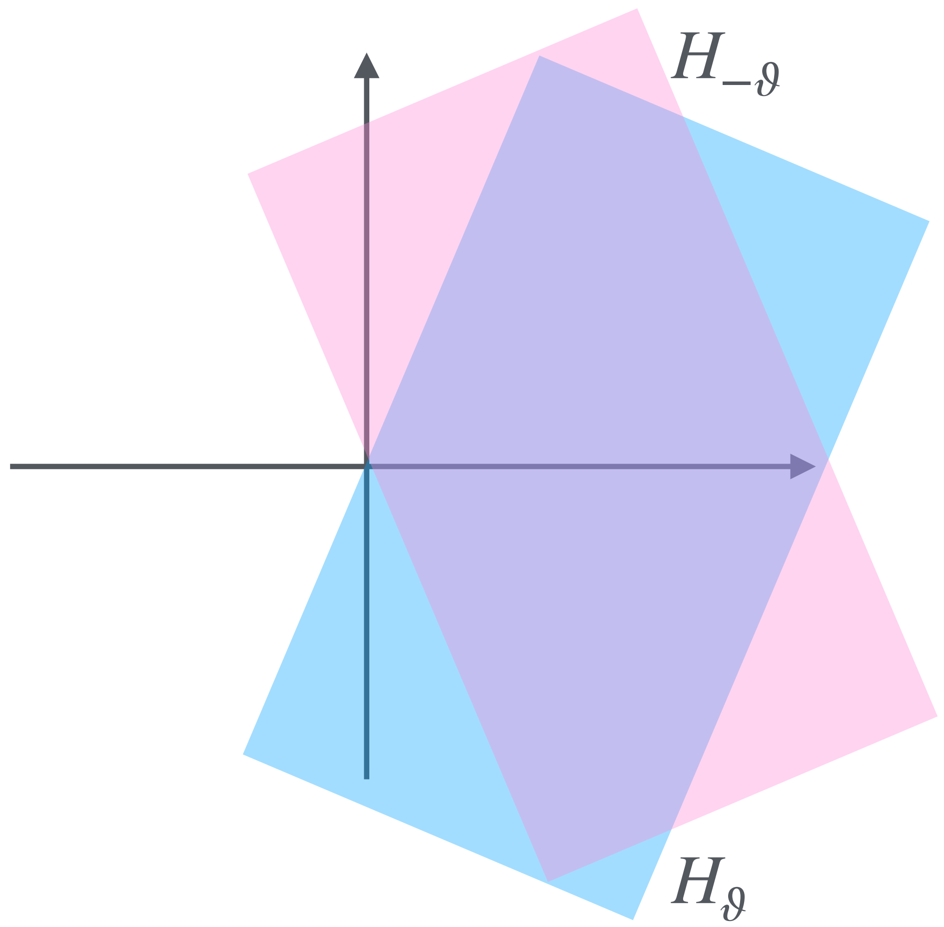
$$\left[\mathscr{L}^{\vartheta} - \mathscr{L}^{-\vartheta}\right]\frac{1}{1-\zeta} = \int_{\mathscr{C}_1} e^{-\zeta/\hbar}\,\frac{1}{1-\zeta}\,d\zeta = -2\pi\mathrm{i}\ \ e^{-1/\hbar}$$

Varying the direction of the Borel-Laplce summation, the sum $\hat{\Phi}$ jumps giving exponentially small corrections

What is the effect of the singularities?

The exponentially small terms can be reconstructed from the divergent series





$$\left[\mathscr{L}^{\vartheta} - \mathscr{L}^{-\vartheta}\right]\frac{1}{1-\zeta} = \int_{\mathscr{C}_1} e^{-\zeta/\hbar}\,\frac{1}{1-\zeta}\,d\zeta = \underbrace{-2\pi \mathrm{i}}_{\text{Residue}}\;\underbrace{e^{-1/\hbar}}_{\text{Singularity}}$$

The resurgence analysis of the Borel transforms already computed these contributions!

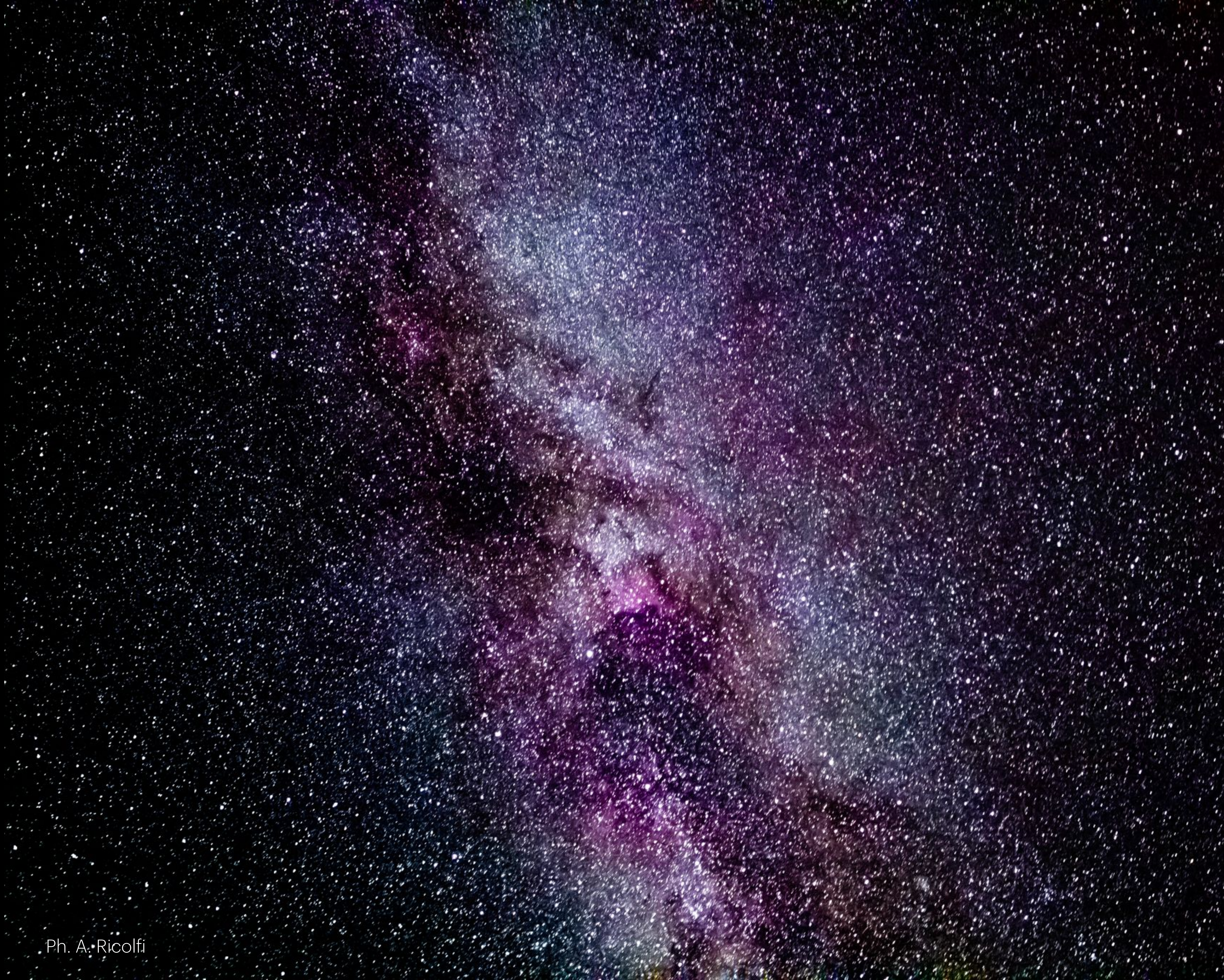
Plan

1. Motivation

2. An overview of resurgence

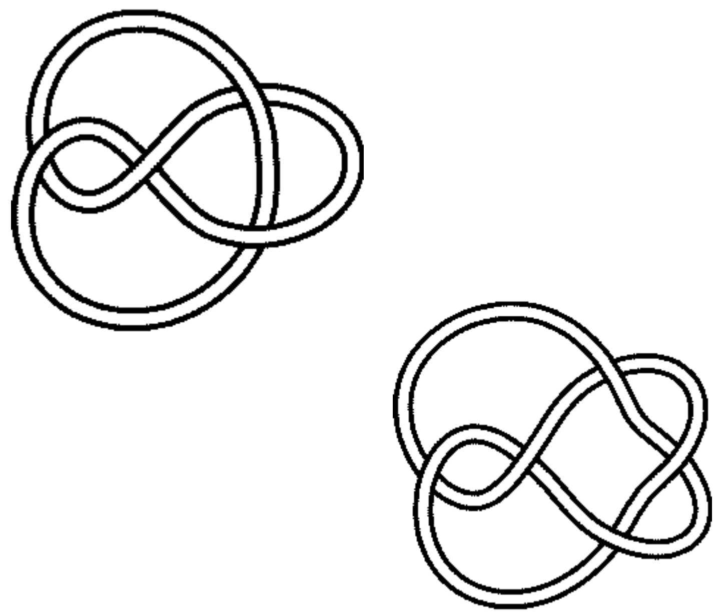
3. **Application to complex Chern-Simons on the complement of a hyperbolic knot**

4. Conclusions



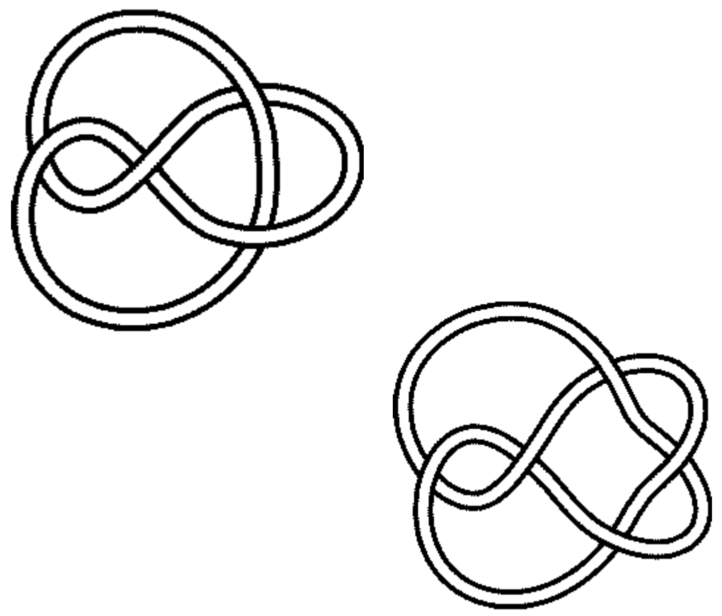
Ph. A. Ricolfi

Topological invariants of hyperbolic knots



$K \rightsquigarrow$ topological invariants [Jones, Kashaev]

The volume conjecture

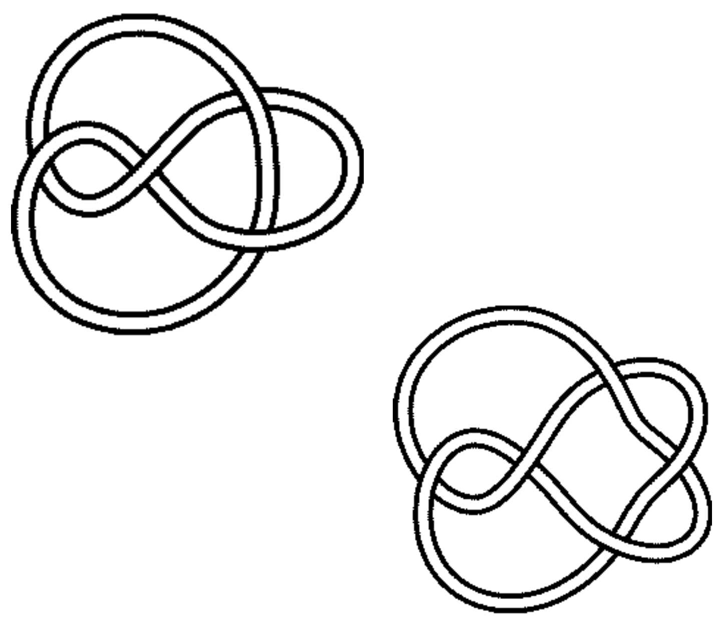


K $\rightsquigarrow$ topological invariants [Jones, Kashaev]

Volume conjecture [Kashaev]

$$\frac{2\pi \log |\langle K \rangle_N|}{N} \sim_{N\to\infty} \mathrm{vol}(S^3 \backslash K)$$

Complex Chern-Simon theory on the complement of an hyperbolic knots



$K \rightsquigarrow$ topological invariants [Jones, Kashaev]

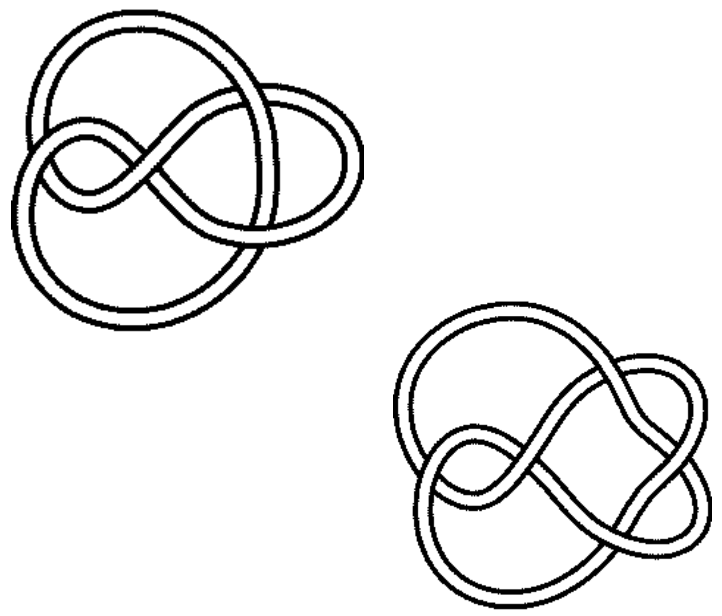
"Find an intrinsically three-dimensional definition of the Jones polynomial of knot theory"

[Atiyah]

SU(2) Chern-Simons on $S^3 \backslash K$

[Witten]

Complex Chern-Simon theory on the complement of an hyperbolic knots



$K \rightsquigarrow$ topological invariants [Jones, Kashaev]

"Find an intrinsically three-dimensional definition of the Jones polynomial of knot theory"

[Atiyah]

$\mathrm{SU}(2)$ Chern-Simons on $S^3 \backslash K$

[Witten]

Analytic continuation of Chern-Simons theory is divergent, and it is expected to *localise* at $\mathrm{SL}_2(\mathbb{C})$ flat connections [Witten]

Perturbative vs non-perturbative topological invariants of hyperbolic knots

Dimofte-Garoufalidis perturbative invariants [Dimofte-Garoufalidis, Garoufalidis-Strozer-Wheeler]

$$\tilde{\Upsilon}(\tau)$$

which represents the all orders asymptotics of Kashaev's invariant, with leading order given by the volume, and it should represent $SL_2(\mathbb{C})$ Chern-Simons on $S^3 \backslash K$

Andersen-Kashaev state integrals [Andersen-Kashaev]

$$I_{0,0}(\tau)$$

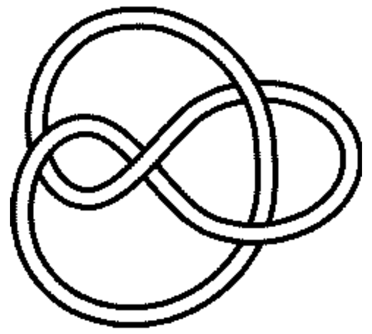
which defines the partition function of a 3D Teichmüller TQFT

Andersen-Kashaev volume conjecture $\quad \tilde{\Upsilon} \xleftarrow[\text{Asymptotics}]{} I_{0,0}$

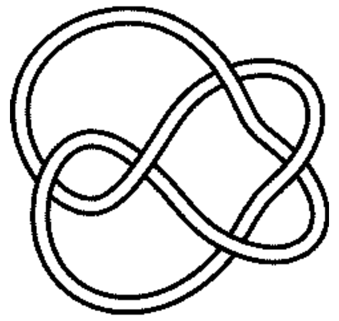
Perturbative vs non-perturbative topological invariants of hyperbolic knots

Dimofte-Garoufalidis perturbative invarianats [Dimofte-Garoufalidis, Garoufalidis-Strozer-Wheeler]

$$\tilde{\Upsilon}(\tau) := \int \tilde{\Psi}(z,\tau)^B \, \mathbf{e}\Big(-\frac{A}{2} z^2 \tau\Big)\, dz$$



$\mathbf{4_1} : (A = 1, B = 2)$



$\mathbf{5_2} : (A = 2, B = 3)$

Andersen-Kashaev state integrals and descendants [Andersen-Kashaev]

$$I_{m,\ell}(\tau) := \int_{\mathcal{J}_{\ell,\tau}} \underbrace{\Phi((z-\ell)\tau;\tau)^B}_{\text{Faddeev's dilogarithm}} \, \mathbf{e}\Big(\frac{A}{2} z(z\tau + \tau + 1) + mz\tau\Big) dz\,,$$

where $m = 0, \ldots, A-1$ et $\ell \in \mathbb{Z}$

$I_{0,0}(\tau)$ is the AK state integral

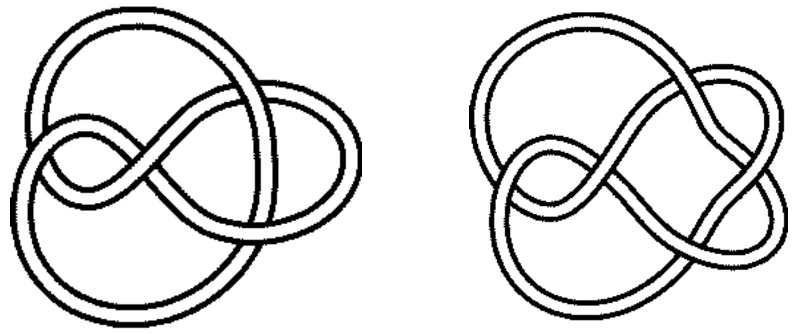
$$\text{Divergent series} \quad \tilde{\Psi} \; \underset{\text{Asymptotics}}{\overset{\text{Borel-Laplace sum}}{\rightleftarrows}} \; \Phi \quad \text{Analytic function}$$

[Kashaev-Garoufalidis]

Perturbative vs non-perturbative topological invariants of hyperbolic knots

Dimofte-Garoufalidis perturbative invarianats [Dimofte-Garoufalidis, Garoufalidis-Strozer-Wheeler]

$$\tilde{\Upsilon}(\tau) := \int \tilde{\Psi}(z,\tau)^B \, \mathbf{e}\left(-\frac{A}{2}z^2\tau\right) dz$$



$\mathbf{4_1}$: $(A=1, B=2)$ $\mathbf{5_2}$: $(A=2, B=3)$

Andersen-Kashaev state integrals and descendants [Andersen-Kashaev]

$$I_{m,\ell}(\tau) := \int_{\mathcal{J}_{\ell,\tau}} \underbrace{\Phi((z-\ell)\tau;\tau)}_{\text{Faddeev's dilogarithm}}{}^B \, \mathbf{e}\left(\frac{A}{2}z(z\tau+\tau+1)+mz\tau\right) dz\,,$$

where $m = 0,\ldots,A-1$ et $\ell \in \mathbb{Z}$

$I_{0,0}(\tau)$ is the AK state integral

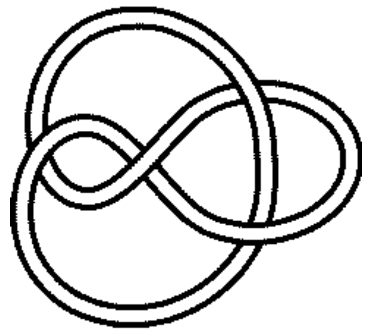
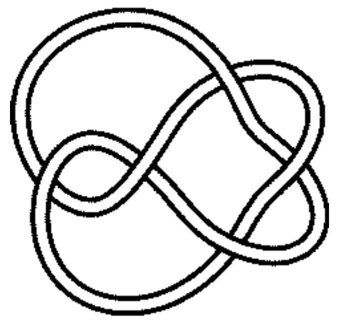
$$\text{Divergent series} \quad \tilde{\Psi} \;\underset{\text{Asymptotics}}{\overset{\text{Borel-Laplace sum}}{\rightleftarrows}}\; \Phi \quad \text{Analytic function}$$

Andersen-Kashaev volume conjecture: $\tilde{\Upsilon} \xleftarrow[\text{Asymptotics}]{} I_{0,0}$

Perturbative vs non-perturbative topological invariants of hyperbolic knots

Dimofte-Garoufalidis perturbative invarianats [Dimofte-Garoufalidis, Garoufalidis-Strozer-Wheeler]

$$\tilde{\Upsilon}(\tau) := \int \tilde{\Psi}(z,\tau)^B \,\mathbf{e}\Big(-\frac{A}{2}z^2\tau\Big)\, dz$$

$\mathbf{4_1} : (A=1, B=2)$ $\mathbf{5_2} : (A=2, B=3)$

Andersen-Kashaev state integrals and descendants
[Andersen-Kashaev]

$$I_{m,\ell}(\tau) := \int_{\mathcal{J}_{\ell,\tau}} \underbrace{\Phi((z-\ell)\tau;\tau)}_{\text{Faddeev's dilogarithm}}{}^B \,\mathbf{e}\Big(\frac{A}{2}z(z\tau+\tau+1)+mz\tau\Big)dz\,,$$

where $m = 0,\ldots,A-1$ et $\ell \in \mathbb{Z}$

$I_{0,0}(\tau)$ is the AK state integral

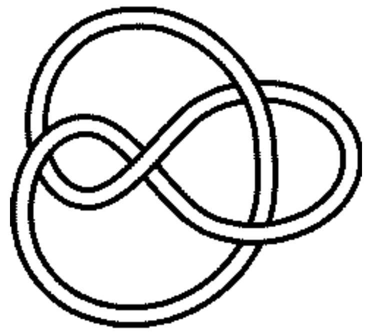
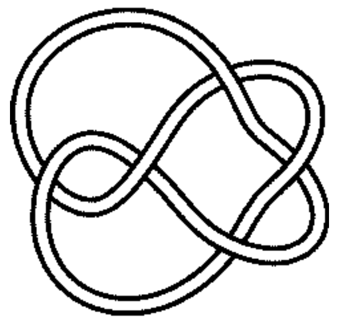
Divergent series $\tilde{\Psi} \underset{\text{Asymptotics}}{\overset{\text{Borel-Laplace sum}}{\rightleftarrows}} \Phi$ Analytic function

$\tilde{\Upsilon} \underset{\text{Asymptotics}}{\overset{\text{Borel-Laplace sum}}{\rightleftarrows}} I_{0,0}$ **?**

Perturbative vs non-perturbative topological invariants of hyperbolic knots

Dimofte-Garoufalidis perturbative invarianats [Dimofte-Garoufalidis, Garoufalidis-Strozer-Wheeler]

$$\tilde{\Upsilon}(\tau) := \int \tilde{\Psi}(z,\tau)^B \,\mathbf{e}\Big(-\frac{A}{2}z^2\tau\Big)\, dz$$

$\mathbf{4_1} : (A=1, B=2)$ $\quad$ $\mathbf{5_2} : (A=2, B=3)$

Andersen-Kashaev state integrals and descendants [Andersen-Kashaev]

$$I_{m,\ell}(\tau) := \int_{\mathcal{J}_{\ell,\tau}} \underbrace{\Phi((z-\ell)\tau;\tau)}_{\text{Faddeev's dilogarithm}}{}^B \,\mathbf{e}\Big(\frac{A}{2}z(z\tau+\tau+1)+mz\tau\Big) dz\,,$$

where $m = 0,\ldots,A-1$ et $\ell \in \mathbb{Z}$

$I_{0,0}(\tau)$ is the AK state integral

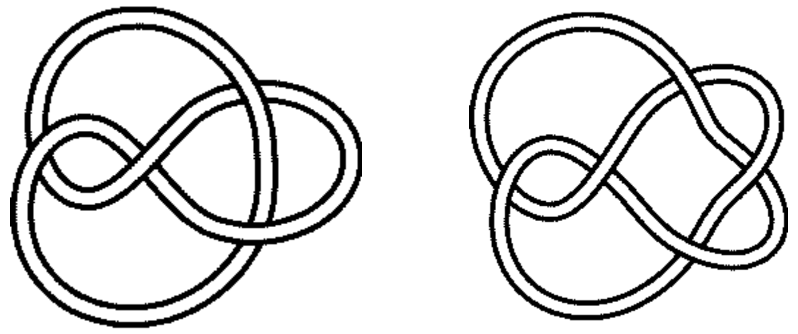
Divergent series $\tilde{\Psi} \underset{\text{Asymptotics}}{\overset{\text{Borel-Laplace sum}}{\rightleftarrows}} \Phi$ Analytic function

Conjecture [Garoufalidis-Gu-Mariño] The Borel sum of $\tilde{\Upsilon}$ is a linear combination of $I_{m,\ell}$

Perturbative vs non-perturbative topological invariants of hyperbolic knots

Dimofte-Garoufalidis perturbative invarianats [Dimofte-Garoufalidis, Garoufalidis-Strozer-Wheeler]

$$\tilde{\Upsilon}(\tau) := \int \tilde{\Psi}(z,\tau)^B \, \mathbf{e}\Big(-\frac{A}{2}z^2\tau\Big)\, dz$$



$\mathbf{4_1}$: $(A = 1, B = 2)$ $\mathbf{5_2}$: $(A = 2, B = 3)$

Andersen-Kashaev state integrals and descendants [Andersen-Kashaev]

$$I_{m,\ell}(\tau) := \int_{\mathcal{J}_{\ell,\tau}} \Phi((z-\ell)\tau;\tau)^B \, \mathbf{e}\Big(\frac{A}{2}z(z\tau+\tau+1)+mz\tau\Big)dz\,,$$

Faddeev's dilogarithm

where $m = 0,\ldots, A-1$ et $\ell \in \mathbb{Z}$

$I_{0,0}(\tau)$ is the AK state integral

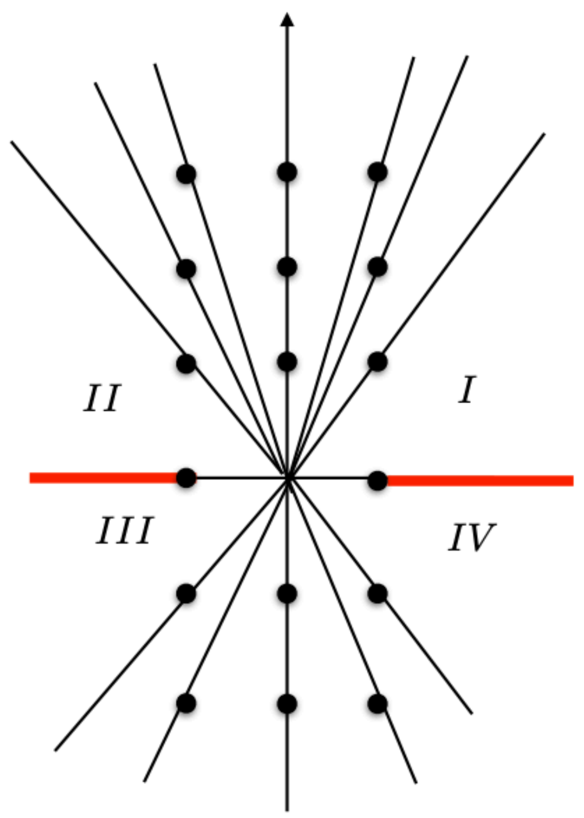
Divergent series $\tilde{\Psi} \underset{\text{Asymptotics}}{\overset{\text{Borel-Laplace sum}}{\rightleftarrows}} \Phi$ Analytic function

Theorem [VF-Wheeler] The Borel sum of $\tilde{\Upsilon}$ is a linear combination of $I_{m,\ell}$ for the 4_1 and 5_2 knots

Resurgence of the DG perturbative invariants

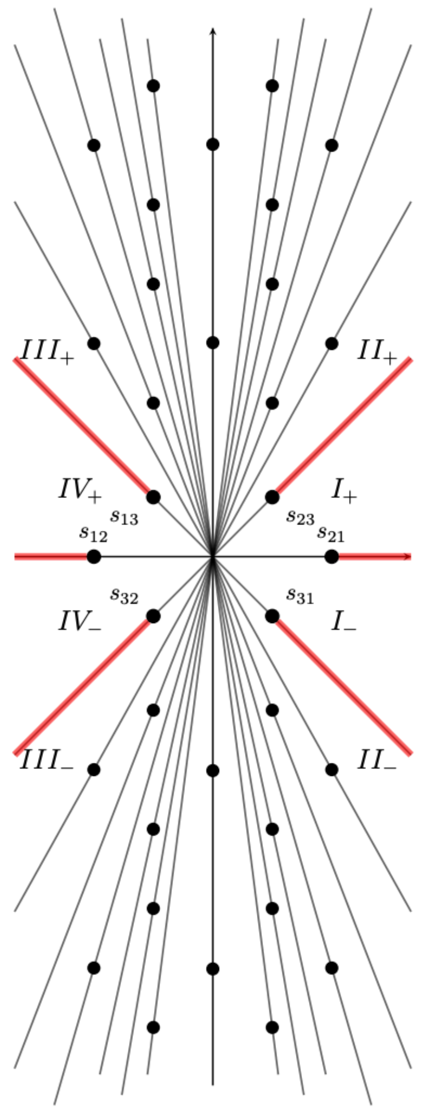
- The singularities in the Borel plane are organized in a **peacock pattern**, and they are located at the **critical values of the Chern-Simons functional** [Garoufalidis-Gu-Mariño]

ζ-plane



4_1

ζ-plane



5_2

- The **Stokes constants** can be computed by solving a q-difference equation [Garoufalidis-Gu-Mariño]

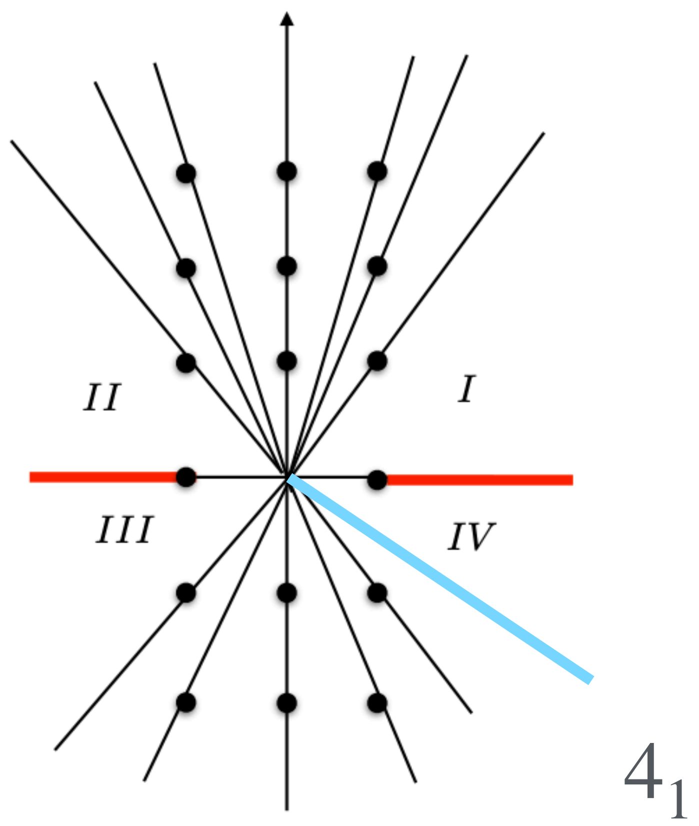
$$M_I(q) = \begin{pmatrix} 1-q-2q^2-2q^3-2q^4 & -1+2q+3q^2+2q^3+q^4 \\ 1-7q-14q^2-8q^3-2q^4 & 1+10q+15q^2-2q^3-19q^4 \end{pmatrix} + O(q^5)$$

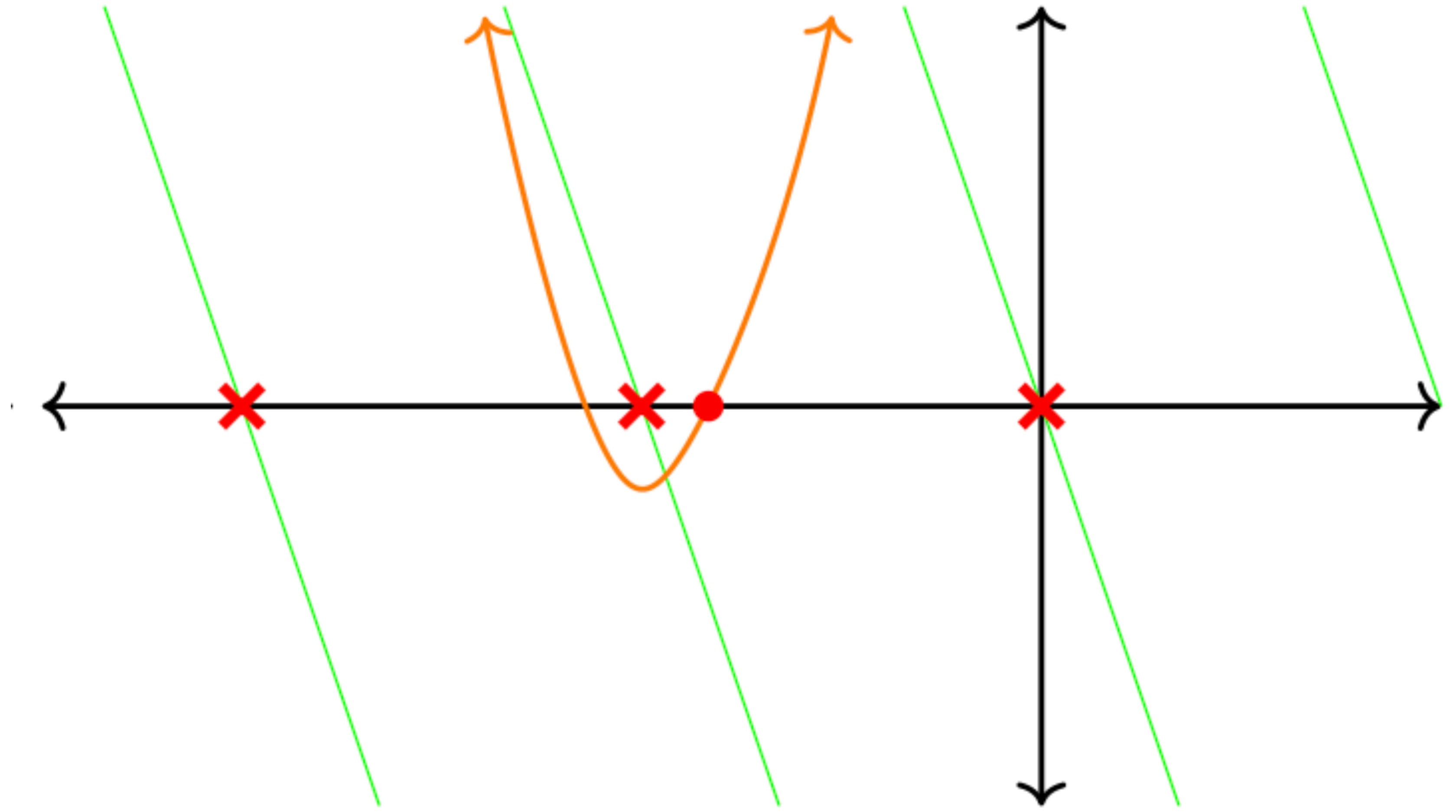
$$M_{I_+}(q) = \begin{pmatrix} -1-q^2 & 2+3q^2 & 1+q+3q^2 \\ -1+3q+3q^2 & 1-6q-3q^2 & -q \\ -\frac{5}{6}+5q-\frac{53}{6}q^2 & -\frac{4}{3}-4q+\frac{77}{2}q^2 & -\frac{1}{6}+\frac{29}{6}q+\frac{55}{2}q^2 \end{pmatrix} + O(q^3)$$

- **Conjecture** [Garoufalidis-Gu-Mariño] The Stokes constants have an interpretation in terms of the **3D index** [Dimofte-Gaiotto-Gukov]

Resummation

The Borel-Laplace sum $\hat{\Upsilon}_{\vartheta}$ of the DG invariants $\tilde{\Upsilon}$ is given by a **thimble integral**, whose integration contour can be described **algorithmically** and represented by the **integration contour** of the **state integrals** $I_{m,\ell}$

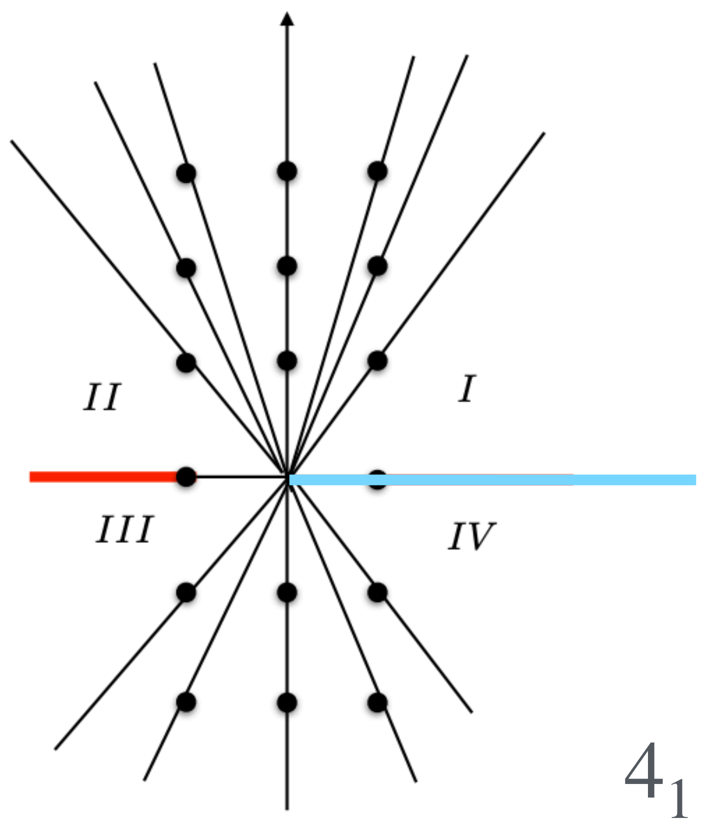


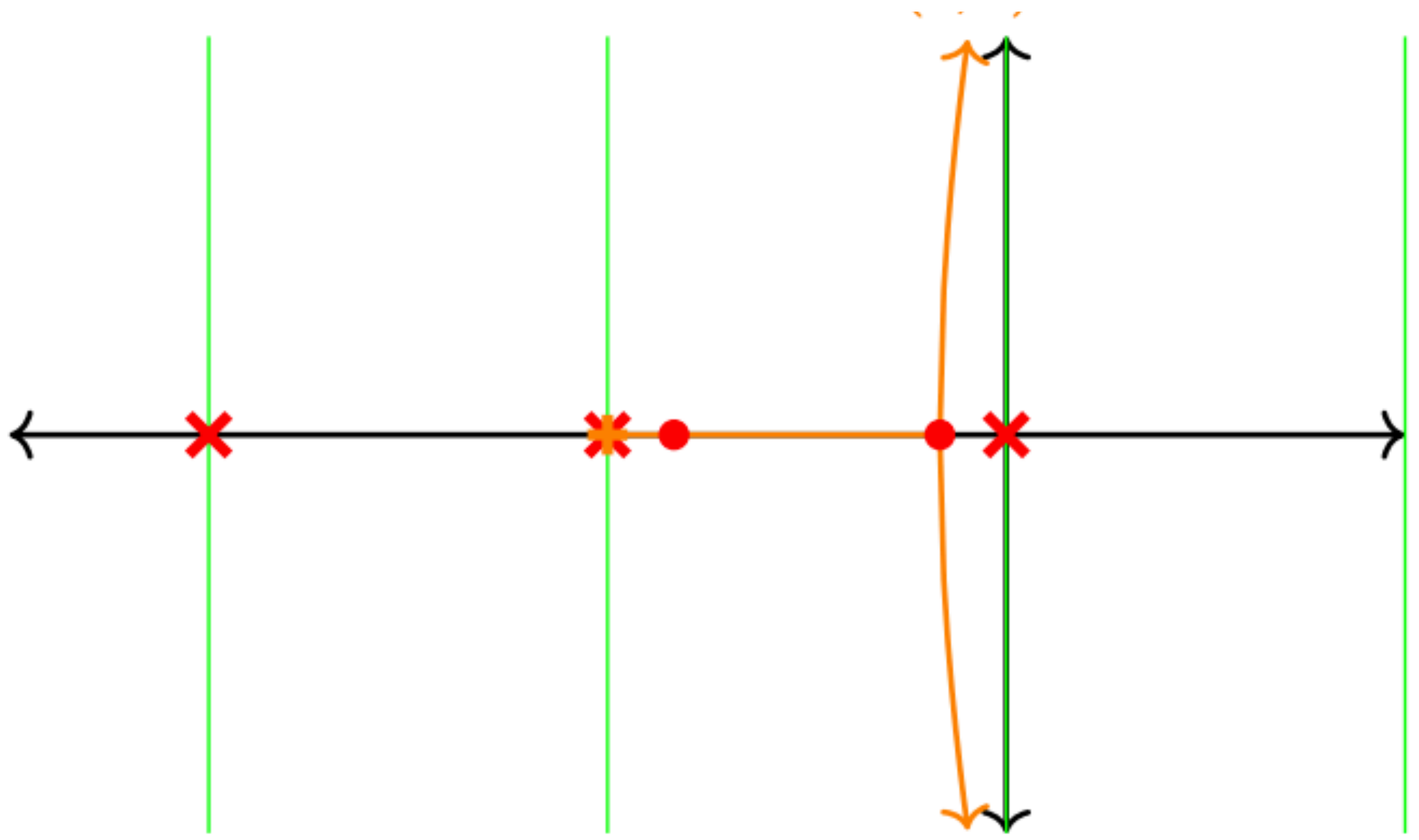


$$\hat{\Upsilon}_{\vartheta} = I_{0,0} + q^2 I_{2,-1} = I_{0,0} + I_{1,0}$$

Resummation

The Borel-Laplace sum $\hat{\Upsilon}_\vartheta$ of the DG invariants $\tilde{\Upsilon}$ is given by a **thimble integral**, whose integration contour can be described **algorithmically** and represented by the **integration contour** of the **state integrals** $I_{m,\ell}$

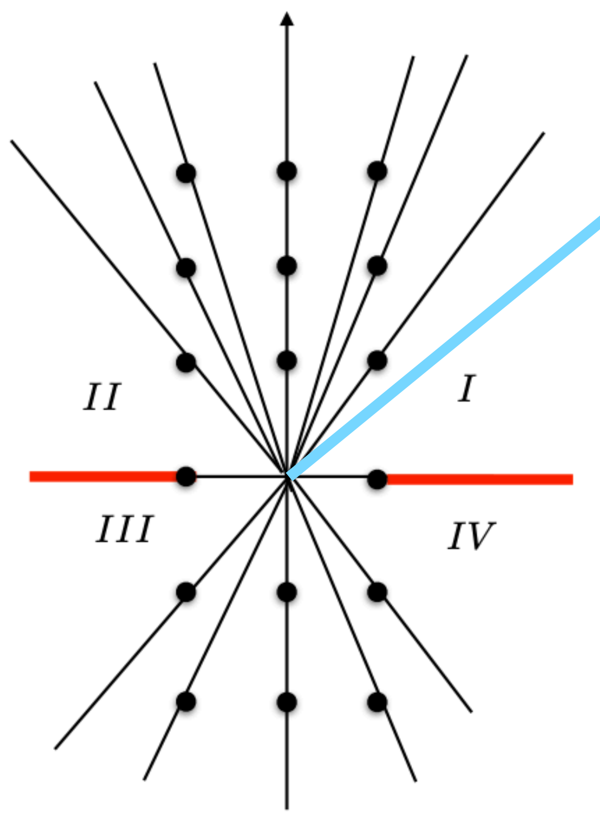




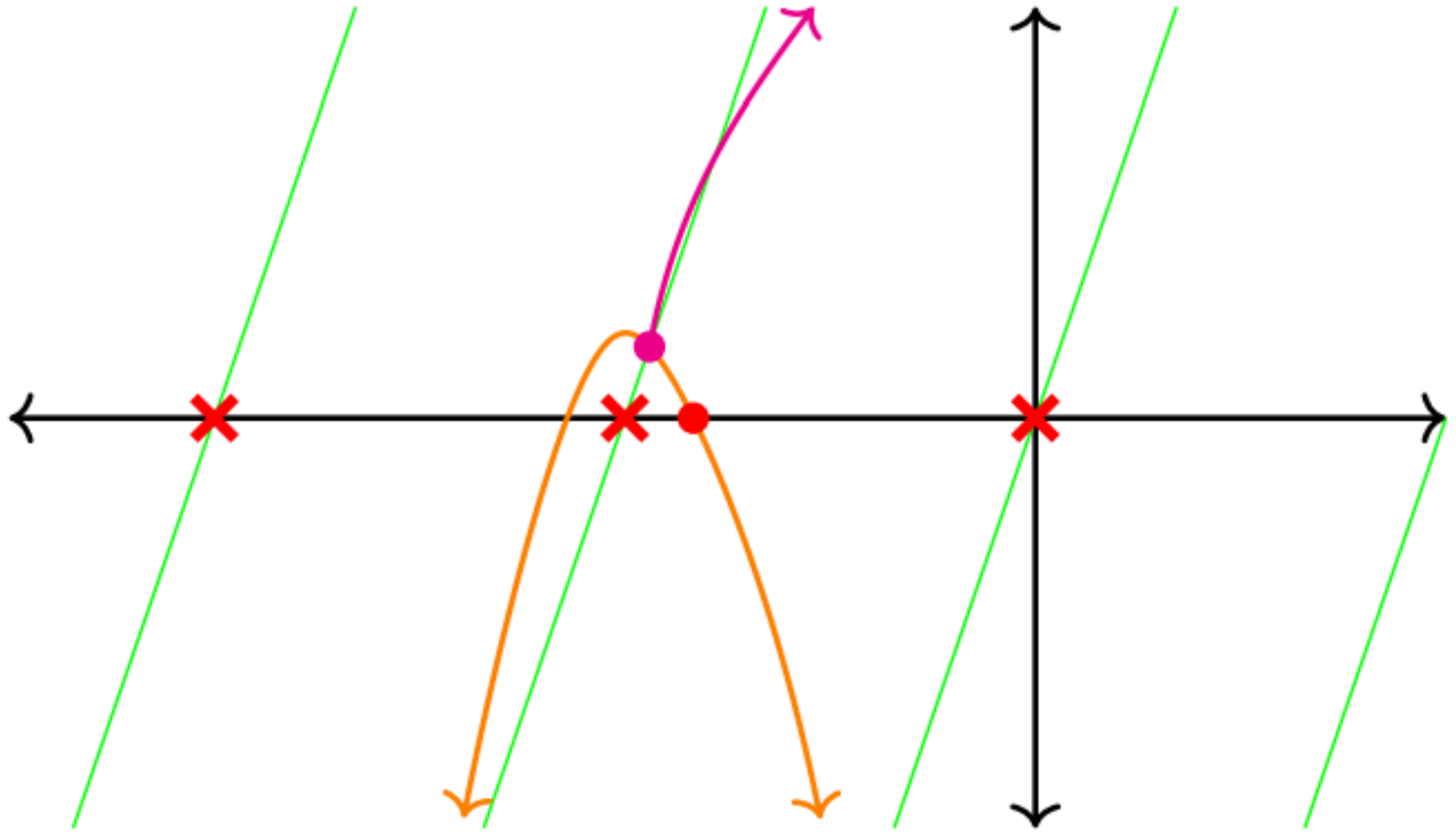
Saddle connection, the resummation is not defined!

Resummation

The Borel-Laplace sum $\hat{\Upsilon}_\vartheta$ of the DG invariants $\tilde{\Upsilon}$ is given by a **thimble integral**, whose integration contour can be described **algorithmically** and represented by the **integration contour** of the **state integrals** $I_{m,\ell}$



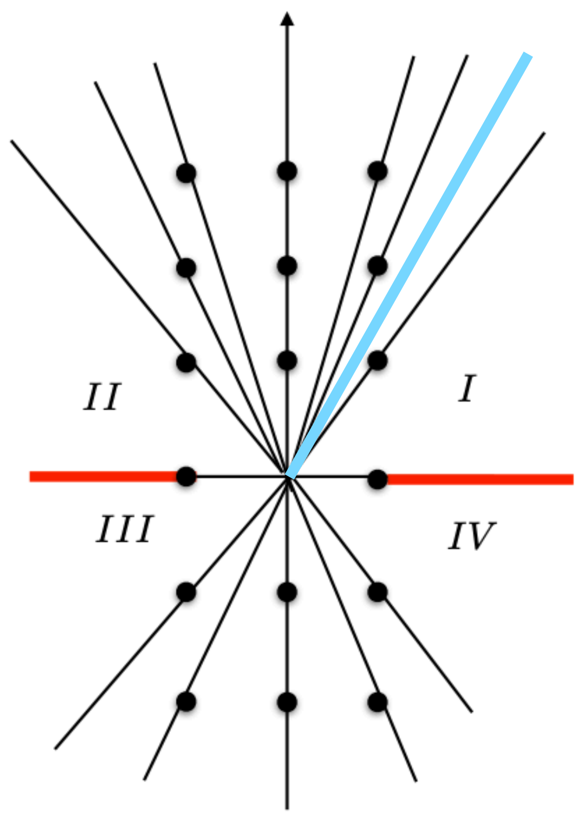
4_1



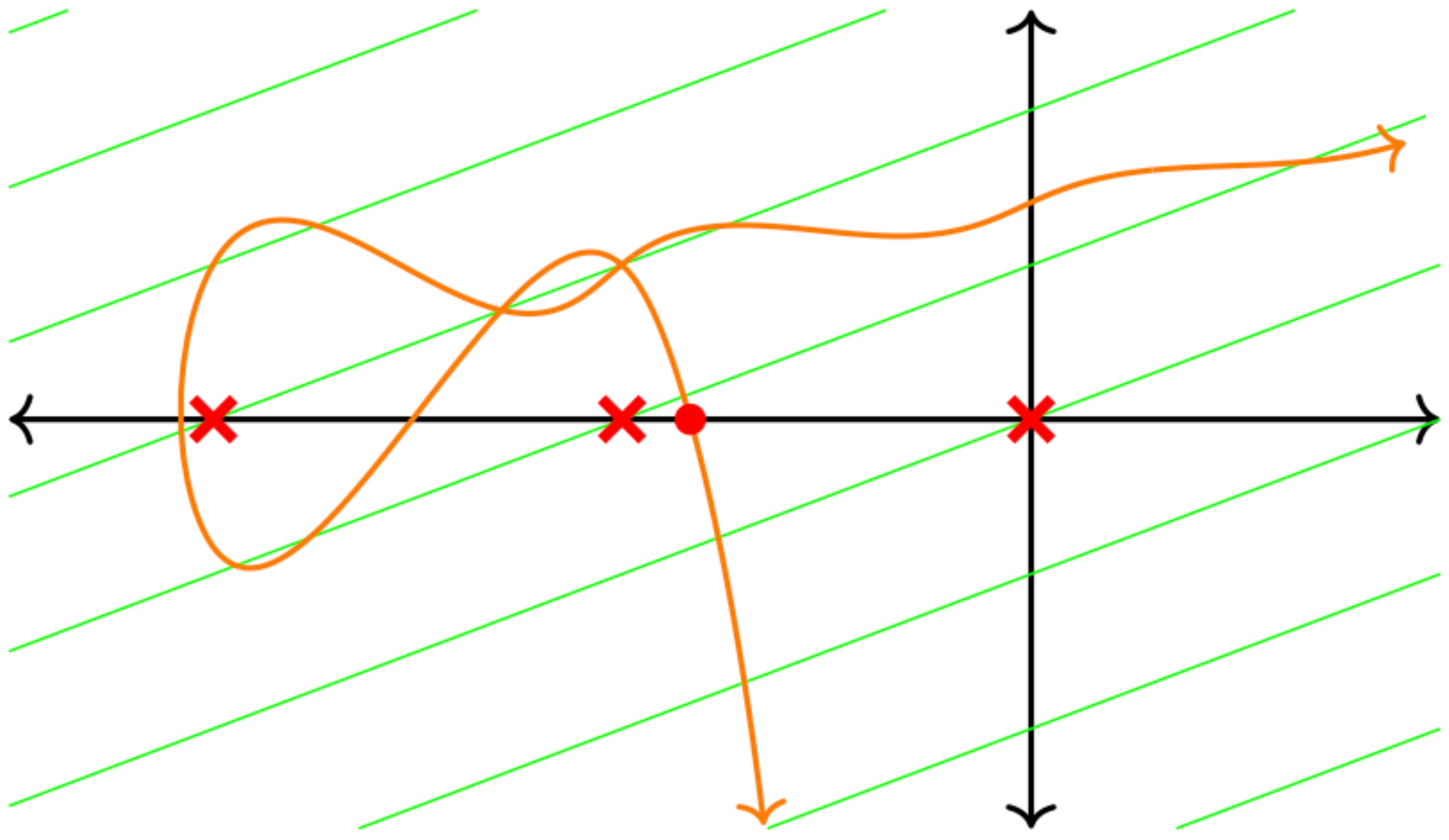
$$\hat{\Upsilon}_\vartheta = -I_{0,0} + I_{0,-1} = -I_{0,0} - I_{-1,0}$$

Resummation

The Borel-Laplace sum $\hat{\Upsilon}_\vartheta$ of the DG invariants $\tilde{\Upsilon}$ is given by a **thimble integral**, whose integration contour can be described **algorithmically** and represented by the **integration contour** of the **state integrals** $I_{m,\ell}$



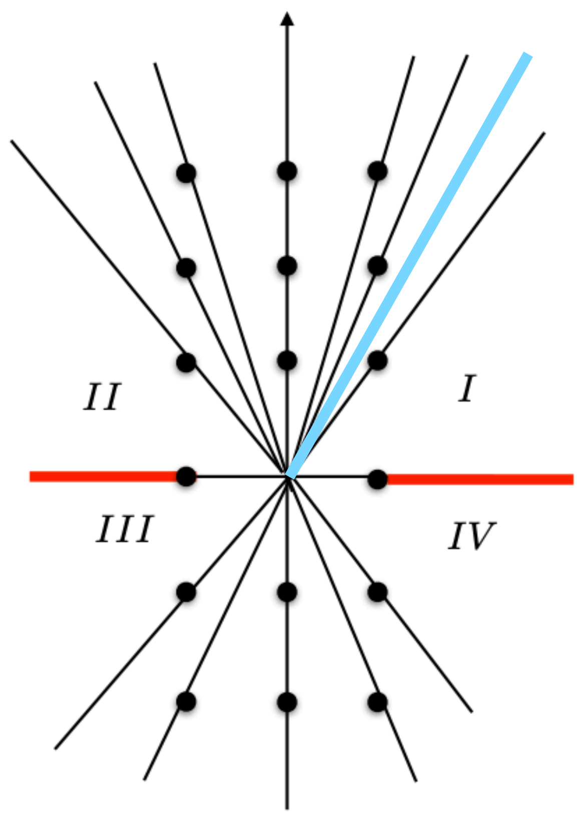
4_1



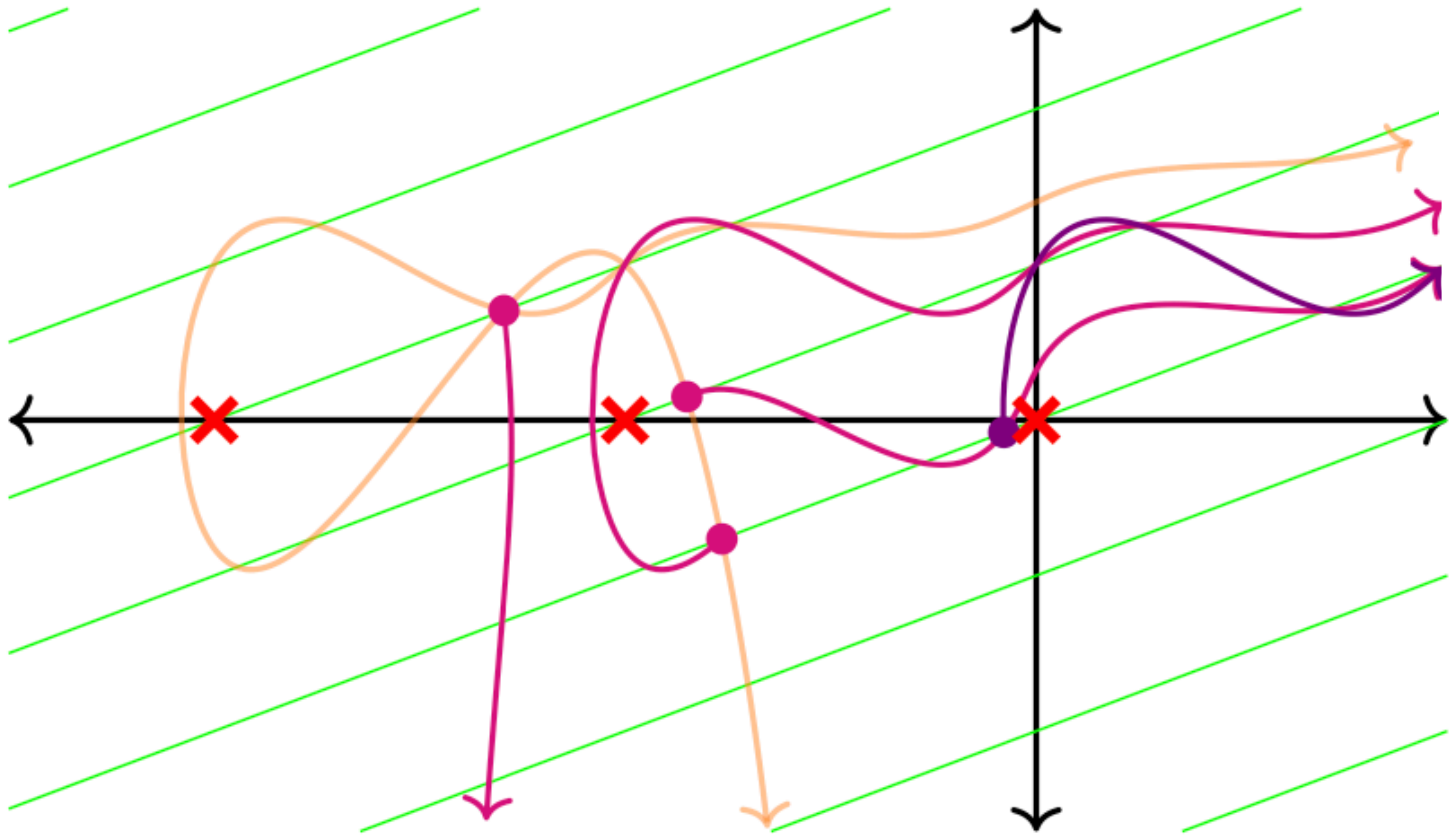
$$\hat{\Upsilon}_\vartheta = -I_{0,0} + I_{0,-1} - q^4 I_{2,-2} + 2q^2 I_{1,-1} + 2q^2 I_{1,-1} - 4q I_{0,0}$$
$$= -I_{0,0} - I_{-1,0} - 9q I_{0,0}$$

Resummation

The Borel-Laplace sum $\hat{\Upsilon}_\vartheta$ of the DG invariants $\tilde{\Upsilon}$ is given by a **thimble integral**, whose integration contour can be described **algorithmically** and represented by the **integration contour** of the **state integrals** $I_{m,\ell}$



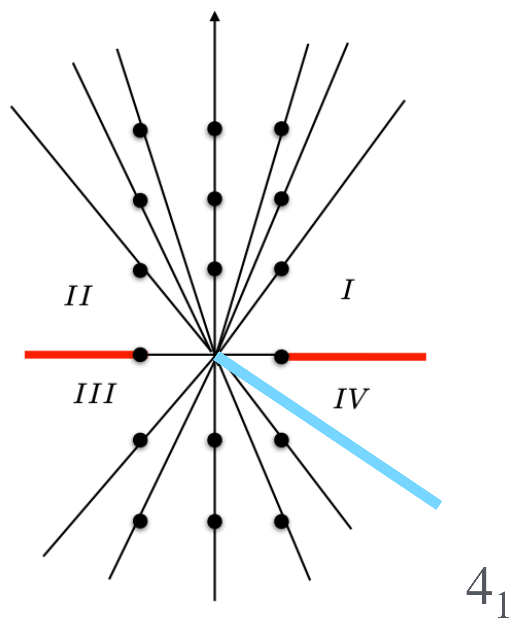
4_1

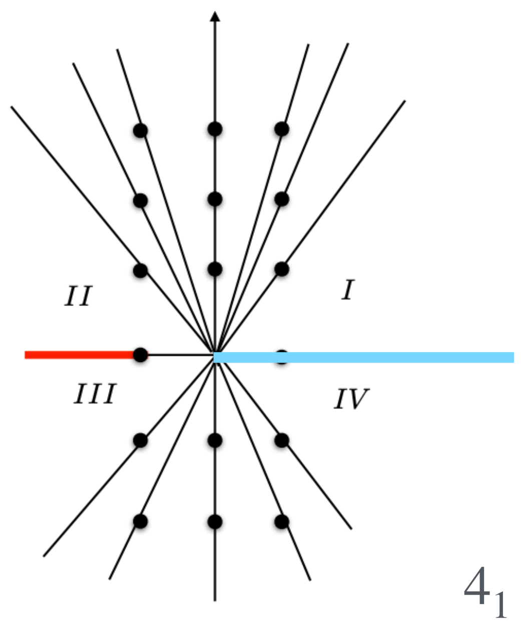


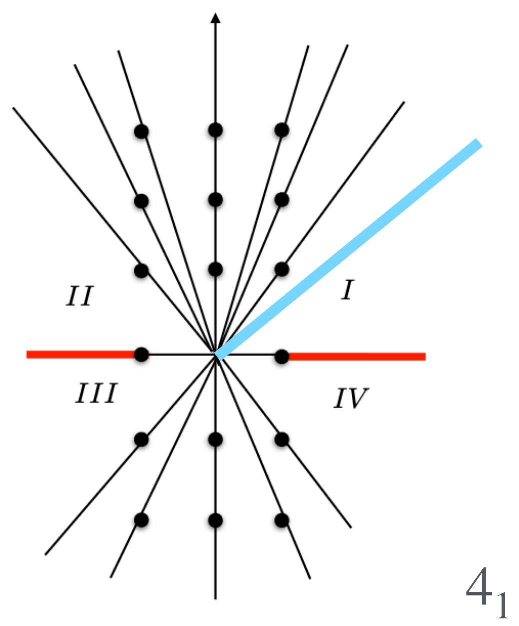
$$\hat{\Upsilon}_\vartheta = -I_{0,0} + I_{0,-1} - q^4 I_{2,-2} + 2q^2 I_{1,-1} + 2q^2 I_{1,-1} - 4q I_{0,0}$$
$$= -I_{0,0} - I_{-1,0} - 9q I_{0,0}$$

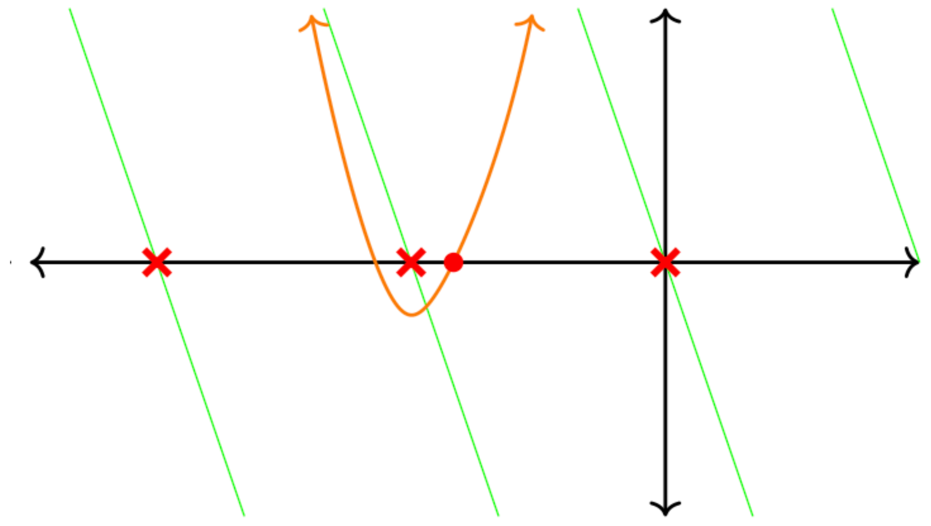
Computing the Stokes constants

The **Stokes constants** are computed by comparing two different Borel-Laplace sums along directions separated by the ray that contains the singularity

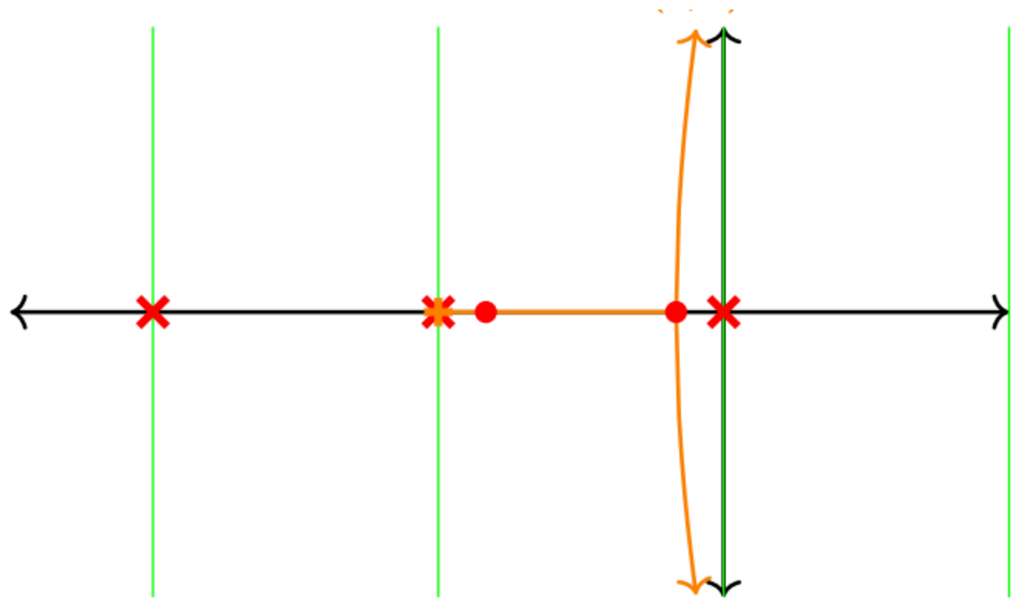


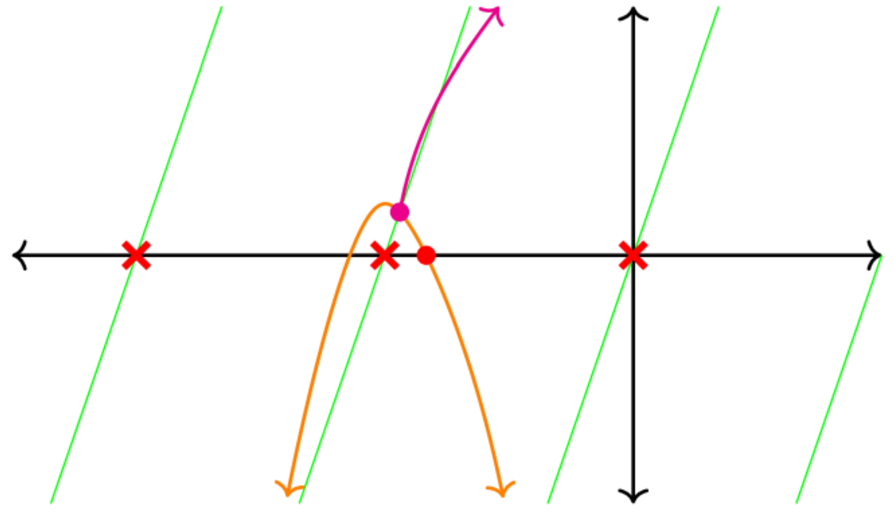






$\hat{\Upsilon}_\vartheta = I_{0,0} + q^2 I_{2,-1} = I_{0,0} + I_{1,0}$





$\hat{\Upsilon}_\vartheta = -I_{0,0} + I_{0,-1} = -I_{0,0} - I_{-1,0}$

$$I_{0,0} + I_{1,0} - (-I_{0,0} - I_{-1,0}) = 3I_{0,0} \Rightarrow S = 3q$$

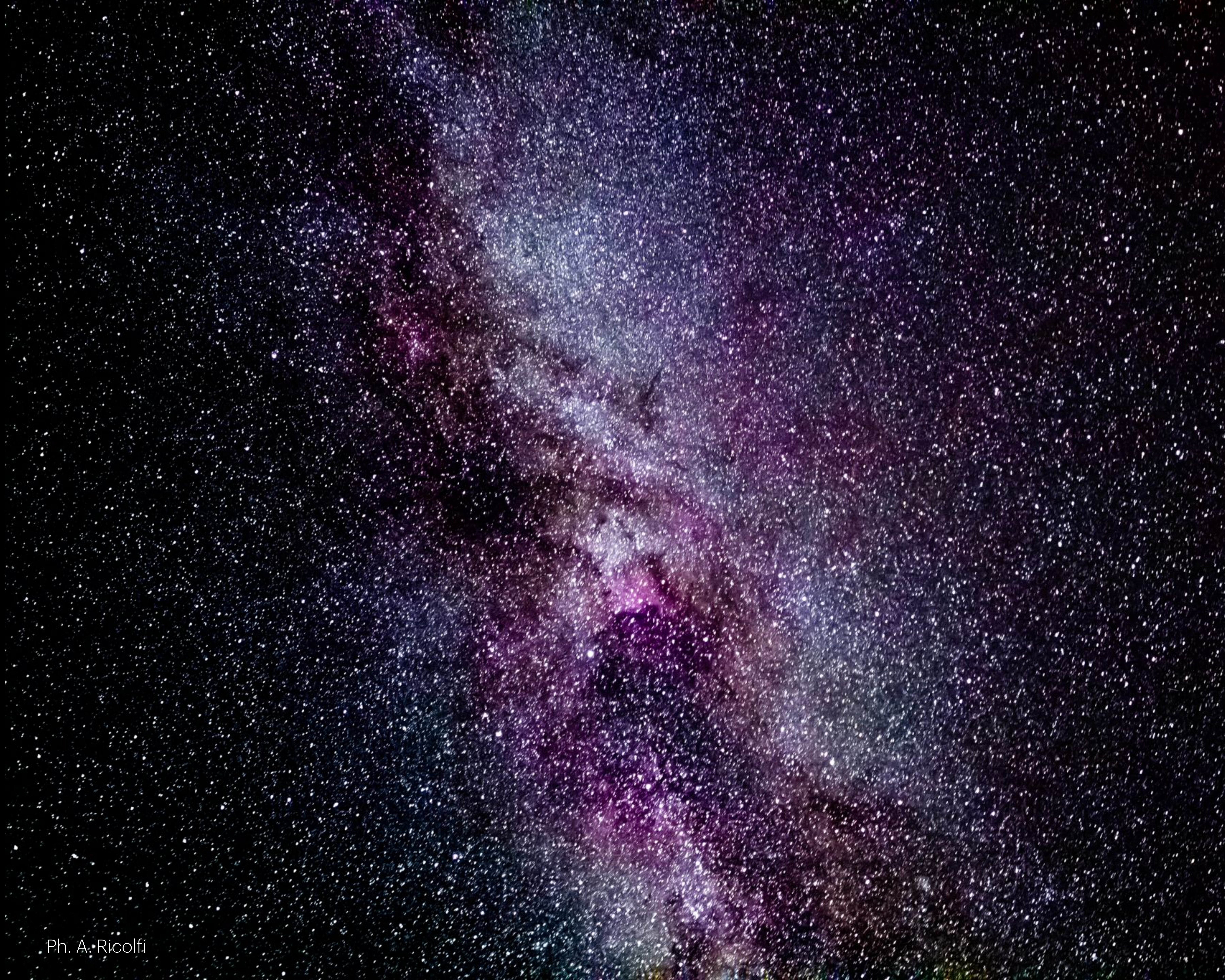
Plan

1. Motivation

2. A overview of resurgence

3. Application to complex Chern-Simons on the complement of an hyperbolic knot

4. Conclusions



Ph. A. Ricolfi

In progress with Andersen, Kontsevich and Wheeler

- Generalization of our result to higher-dimensional integrals, and for a more general class of integrals

$$\int g(x, \hbar)\, dx$$

with $g(x;\hbar) \ \sim_{\hbar\to 0}\ e^{-f(x)/\hbar} \sum_{k\geq 0} \phi_k(x)\, \hbar^k$ and $f\colon X \to \mathbb{C}$ possibly multi-valued

- **Proof of the 3D index conjecture**: the 3D index is computed by the Picard-Lefschetz formula on a suitable homology theory

Can resurgence fail?

Resurgence computes non-perturbative corrections

Despite being a powerful computational tool, effective in various contexts, resurgence has some limits

- Not enough perturbative coefficients limit the numerical computation
- Some non-perturbative sectors can be missed
    - ✦ Trivial flat connection is not seen from the geometric ones [Garoufalidis-Gu-Mariño-Wheeler]
    - ✦ More examples from QFT


Thank you for your attention