

BPS black holes and generalized error functions

Boris Pioline



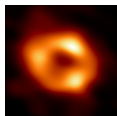
Workshop "Cosmology and High Energy Physics X"
Paroisse Ste Bernadette, Montpellier, 18/09/25

*based on work with Rishi Raj, arXiv:2507.08551
+earlier works with S. Alexandrov, J. Manschot, S. Banerjee, ...*

Introduction

- A central goal for any theory of quantum gravity is to provide a **microscopic explanation** of the **thermodynamical entropy of black holes** in General Relativity [*Bekenstein'72, Hawking'74*]

$$S_{BH} = \frac{A}{4G_N}$$



$$S_{BH} \stackrel{?}{=} \log \Omega$$

- As shown by [*Strominger Vafa'96,...*], String Theory provides a quantitative description in the case of **BPS black holes in vacua with extended SUSY**: at weak string coupling, black hole micro-states arise as **bound states of D-branes** wrapped on cycles of the internal manifold.
- Besides confirming the consistency of string theory as a theory of quantum gravity, this has opened up many fruitful connections with mathematics.

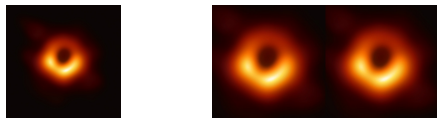
- In the context of type IIA strings compactified on a Calabi-Yau three-fold X , BPS states are described mathematically by **stable objects in the derived category of coherent sheaves** $\mathcal{C} = D^b\text{Coh}X$. The Chern character $\gamma = (\text{ch}_0, \text{ch}_1, \text{ch}_2, \text{ch}_3)$ is identified as the electromagnetic charge, or D6-D4-D2-D0-brane charge.
- The problem becomes a question in **Donaldson-Thomas theory**: for fixed $\gamma \in K(X)$, compute the **generalized DT invariant** $\Omega_Z(\gamma)$ counting **(semi)stable objects** of class γ , and determine its growth as $|\gamma| \rightarrow \infty$.
- Importantly, $\Omega_Z(\gamma)$ depends on the moduli of X , or more generally on a choice of **Bridgeland stability condition** $\sigma \in \text{Stab}\mathcal{C}$. In particular, it can jump on real-codimension one loci known as **walls of marginal stability**. The jump is governed by a universal **wall-crossing formula** [Joyce Song'08, Kontsevich Soibelman'08].

Wall-crossing and black hole bound states

- Walls correspond to loci where the decay $\gamma \rightarrow \sum_i \gamma_i$ is energetically possible. Since $m_i = |Z(\gamma_i)|$, this is possible only when the phases of all $Z(\gamma_i)$ are aligned.
- In the simplest 'primitive' case $\gamma \rightarrow \gamma_1 + \gamma_2$, the jump is

$$\Delta\Omega_\sigma(\gamma_1 + \gamma_2) = \langle \gamma_1, \gamma_2 \rangle \Omega_\sigma(\gamma_1) \Omega_\sigma(\gamma_2)$$

easily reproduced from the SUSY quantum mechanics of the electron-monopole problem (more on this later).



- More generally, the jump can involve an arbitrary number of constituents. The dynamics is complicated, but the index is computable using localization [*Manschot BP Sen'10*].

Modularity of D4-D2-D0 indices I

- Viewing Type IIA string theory as the reduction of M-theory on a circle, allows to make very non-trivial physical predictions about generalized DT invariants. E.g. the GW/DT relation of [MNOP'03].
- In particular, D4-D2-D0 black holes turn out to be black strings in disguise, obtained by wrapping an M5-brane on a divisor $\mathcal{D}$. This indicates that suitable generating series of rank 0 DT invariants should have specific **modular properties** [Maldacena Strominger Witten'97]. This gives very good control on their asymptotic growth, and allows to test agreement with the BH prediction $\Omega_z(\gamma) \simeq e^{S_{BH}(\gamma)}$.
- More precisely, D4-D2-D0 indices occur as Fourier coefficients in the **elliptic genus** $\mathcal{I}(\tau, z^a) = \text{Tr}(-1)^F q^{L_0 - \frac{c_L}{24}} e^{2\pi i q_a z^a}$ of the **two-dimensional superconformal field theory** with $(0, 4)$ SUSY.

Modularity of D4-D2-D0 indices II

- Using the spectral flow symmetry, the elliptic genus has a theta series decomposition

$$\mathcal{I} = \sum_{\mu \in \Lambda^* / \Lambda} h_{p,\mu}(\tau) \Theta(\tau, z^a)$$

where Λ^* / Λ is the finite discriminant group associated to $\Lambda = (H_4(X, \mathbb{Z}), \kappa_{ab} := \kappa_{abc} p^c)$, and

$$h_{p,\mu}(\tau) := \sum_n \bar{\Omega}(0, p, \mu, n) q^{n - \frac{\chi(\mathcal{D})}{24} + \frac{1}{2}\mu^2 - \frac{1}{2}p\mu}$$

- If the SCFT has a **discrete** spectrum, $h_{p,\mu}(\tau)$ must be a vector-valued, weakly holomorphic modular form in the (dual) Weil representation attached to Λ .

Modularity of D4-D2-D0 indices III

- When $\mathcal{D}$ is **very ample** and **irreducible**, there are no walls extending to large volume, so the choice of chamber is irrelevant. The central charges are given by [Maldacena Strominger Witten'97]

$$\begin{cases} c_L = p^3 + c_2(TX) \cdot p = \chi(\mathcal{D}) , \\ c_R = p^3 + \frac{1}{2}c_2(TX) \cdot p = 6\chi(\mathcal{O}_{\mathcal{D}}) \end{cases}$$

Cardy's formula predicts a growth $\Omega(0, p, \beta, n \rightarrow \infty) \sim e^{2\pi\sqrt{p^3 n}}$ in perfect agreement with Bekenstein-Hawking formula !

- Moreover, since the space of vector-valued weakly holomorphic modular form has finite dimension, the full series is completely determined by its **polar coefficients**, with $n + \frac{1}{2}\mu^2 - \frac{1}{2}p\mu < \frac{\chi(\mathcal{D})}{24}$.

Mock modularity of rank 0 DT invariants

- When $\mathcal{D}$ is reducible, the generating series $h_{p^a, \mu_a}(\tau)$ in a suitable ("large volume attractor") chamber is expected to be a **mock modular form of higher depth** [Alexandrov BP Manschot'16-20])
- Namely, there exists explicit, universal **non-holomorphic theta series** $\Theta_n(\{p_i\}, \tau, \bar{\tau})$ such that (ignoring the μ 's for simplicity)

$$\hat{h}_p(\tau, \bar{\tau}) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n \geq 2} p_i} \Theta_n(\{p_i\}, \tau, \bar{\tau}) \prod_{i=1}^n h_{p_i}(\tau)$$

transforms as a modular form. The completed series satisfy the **holomorphic anomaly equation**,

$$\partial_{\bar{\tau}} \hat{h}_p(\tau, \bar{\tau}) = \sum_{p=\sum_{i=1}^{n \geq 2} p_i} \hat{\Theta}_n(\{p_i\}, \tau, \bar{\tau}) \prod_{i=1}^n \hat{h}_{p_i}(\tau, \bar{\tau})$$

- Θ_n and $\widehat{\Theta}_n$ belongs to the class of **indefinite theta series**

$$\vartheta_{\Phi,q}(\tau, \bar{\tau}) = \tau_2^{-\lambda} \sum_{k \in \Lambda + q} \Phi\left(\sqrt{2\tau_2}k\right) e^{-i\pi\tau Q(k)}$$

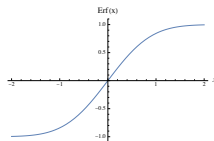
where (Λ, Q) is an even lattice of signature $(r, d-r)$, $q \in \Lambda^*/\Lambda$, $\lambda \in \mathbb{R}$. The series converges if $f(x) \equiv \Phi(x)e^{\frac{\pi}{2}Q(x)} \in L_1(\Lambda \otimes \mathbb{R})$.

- Theorem (Vignéras, 1978): $\{\vartheta_{\Phi,q}, q \in \Lambda^*/\Lambda\}$ transforms as a vector-valued modular form of weight $(\lambda + \frac{d}{2}, 0)$ provided
 - $R(x)f, R(\partial_x)f \in L_2(\Lambda \otimes \mathbb{R})$ for any polynomial $R(x)$ of degree ≤ 2
 - $[\partial_x^2 + 2\pi(x\partial_x - \lambda)] \Phi = 0$ [*]
- The relevant lattice for Θ_n and $\widehat{\Theta}_n$ is $\Lambda = H^2(X, \mathbb{Z})^{\oplus(n-1)}$, with signature $(r, d-r) = (n-1)(1, b_2(X) - 1)$.

Indefinite theta series

- Example 1 (Siegel): $\Phi = e^{\pi Q(x_+)}$, where x_+ is the projection of x on a fixed plane of dimension r , satisfies $[*]$ with $\lambda = -n$. ϑ_Φ is then the usual (non-holomorphic) **Siegel-Narain theta series**.
- Example 2 (Zwegers): In signature $(1, d-1)$, choose C, C' two vectors such that $Q(C), Q(C'), (C, C') > 0$, then

$$\widehat{\Phi}(x) = \operatorname{Erf}\left(\frac{(C, x)\sqrt{\pi}}{\sqrt{Q(C)}}\right) - \operatorname{Erf}\left(\frac{(C', x)\sqrt{\pi}}{\sqrt{Q(C')}}\right)$$



satisfies $[*]$ with $\lambda = 0$. As $|x| \rightarrow \infty$, or if $Q(C) = Q(C') = 0$,

$$\widehat{\Phi}(x) \rightarrow \Phi(x) := \operatorname{sgn}(C, x) - \operatorname{sgn}(C', x)$$

- The theta series $\Theta_2(\{p_1, p_2\})$, $\widehat{\Theta}_2(\{p_1, p_2\})$ fall in this class. The generalization to $n \geq 3$ involves **generalized error functions**.

Generalized error functions I

- Extend the representations

$$\begin{aligned}E_1(x) &= \int_{\mathbb{R}} e^{-\pi(x-x')^2} \operatorname{sign}(x') \, dx' = \operatorname{Erf}(x\sqrt{\pi}), \\M_1(x) &= \frac{i}{\pi} \int_{\mathbb{R}-ix} e^{-\pi z^2 - 2\pi iz} \frac{dz}{z} = -\operatorname{sign}(x) \operatorname{Erfc}(|x|\sqrt{\pi}),\end{aligned}$$

to $\mathbf{x} \in \mathbb{R}^r$, $\mathcal{M} \in \mathbb{R}^{r \times r}$ [Alexandrov Banerjee Manschot BP'16, Nazaroglu'16]

$$\begin{aligned}E_r(\mathcal{M}, \mathbf{x}) &= \int_{\mathbb{R}^r} d^r \mathbf{z} e^{-\pi(\mathbf{x}-\mathbf{z})^T(\mathbf{x}-\mathbf{z})} \prod_{i=1}^r \operatorname{sign}(\mathcal{M}^T \mathbf{z})_i, \\M_r(\mathcal{M}; \mathbf{x}) &= \left(\frac{i}{\pi}\right)^r |\det \mathcal{M}|^{-1} \int_{\mathbb{R}^r - i\mathbf{x}} d^r \mathbf{z} \frac{e^{-\pi \mathbf{z}^T \mathbf{z} - 2\pi i \mathbf{z}^T \mathbf{x}}}{\prod_{i=1}^r (\mathcal{M}^{-1} \mathbf{z})_i}\end{aligned}$$

Generalized error functions II

- Both E_r and M_r are annihilated by Vignéras operator $\partial_{\mathbf{x}}^2 + 2\pi\mathbf{x}\partial_{\mathbf{x}}$.
- $E_r(\mathcal{M}, \mathbf{x})$ is a C^∞ function of $\mathbf{x}$, which asymptotes to $\prod_{i=1}^r \text{sgn}(\mathcal{M}^T \mathbf{x})_i$ as $|\mathbf{x}| \rightarrow \infty$
- $M_r(\mathcal{M}, \mathbf{x})$ is a C^∞ function away from the hyperplanes $(\mathcal{M}^{-1} \mathbf{x})_i = 0$, exponentially suppressed as $|\mathbf{x}| \rightarrow \infty$
- Both E_r and M_r are invariant under rescaling the columns of $\mathcal{M}$ by arbitrary positive factors or permuting them, and under rotating $(\mathcal{M}, \mathbf{x}) \mapsto (O\mathcal{M}, O\mathbf{x})$ by $O \in O(r)$.

Generalized error functions III

- For any collection $\mathcal{V} = (\mathbf{v}_1, \dots, \mathbf{v}_r)$ of r vectors in $\mathbb{R}^d$ with fixed quadratic form $\mathcal{Q}$ of signature $(r, d - r)$, such that the Gram matrix $\mathcal{V}^T \mathcal{Q} \mathcal{V}$ is positive definite, define the **boosted error functions**

$$\begin{aligned}\Phi_r^E(\mathcal{Q}, \{\mathbf{v}_i\}, \mathbf{x}) &= E_r(\mathcal{B} \mathcal{Q} \mathcal{V}, \mathcal{B} \mathcal{Q} \mathbf{x}) \\ \Phi_r^M(\mathcal{Q}, \{\mathbf{v}_i\}, \mathbf{x}) &= M_r(\mathcal{B} \mathcal{Q} \mathcal{V}, \mathcal{B} \mathcal{Q} \mathbf{x})\end{aligned}$$

where $\mathcal{B}$ is an orthonormal basis of $\langle \mathcal{V} \rangle$, $\mathcal{B} \mathcal{Q} \mathcal{B}^T = 1$.

- $\Phi_r^E(\mathcal{Q}, \mathcal{V}, \mathbf{x})$ is a C^∞ function of $\mathbf{x}$ which asymptotes to $\text{sgn}(\mathcal{V}^T \mathcal{Q} \mathbf{x})$ as $|\mathbf{x}| \rightarrow \infty$. $\Phi_r^M(\mathcal{Q}, \mathcal{V}, \mathbf{x})$ is exponentially suppressed as $|\mathbf{x}| \rightarrow \infty$.
- Φ_r^E can be expressed in terms of Φ_r^M , and vice-versa,

$$\Phi_r^E(\mathcal{Q}, \{\mathbf{v}_i\}, \mathbf{x}) = \sum_{\mathcal{I} \subset \{1, \dots, r\}} \Phi_{|\mathcal{I}|}^M(\mathcal{Q}, \{\mathbf{v}_i\}_{i \in \mathcal{I}}, \mathbf{x}) \prod_{j \notin \mathcal{I}} \text{sign}(\mathbf{v}_{j \perp \mathcal{I}} \mathcal{Q} \mathbf{x}),$$

where $\mathbf{v}_{j \perp \mathcal{I}}$ is the projection of $\mathbf{v}_j$ orthogonal to the vectors $\mathbf{v}_{i \in \mathcal{I}}$.

Multi-black hole quantum mechanics I

- Consider the Lagrangian with 4 real supercharges

$$L = \sum_{i=1}^n \frac{m_i}{2} \left(\dot{\vec{x}}_i^2 + D_i^2 + 2i\bar{\lambda}_i \dot{\lambda}_i \right) + \sum_{i=1}^n (-U_i D_i + \vec{A}_i \cdot \dot{\vec{x}}_i) + \sum_{i,j=1}^n \vec{\nabla}_i U_j \cdot \bar{\lambda}_i \vec{\sigma} \lambda_j$$

where

$$U_i = -\frac{1}{2} \left(\sum_{j \neq i} \frac{\gamma_{ij}}{|\vec{x}_i - \vec{x}_j|} - c_i \right), \quad \vec{\nabla}_i U_j = \frac{1}{2} (\vec{\nabla}_i \times \vec{A}_j + \vec{\nabla}_j \times \vec{A}_i).$$

- Eliminating the auxiliary fields D_i , one generates a potential

$$V = \sum_{i=1}^n \frac{U_i^2}{2m_i}.$$

Multi-black hole quantum mechanics II

- Supersymmetric ground states satisfy Denef's equations:

$$\sum_{j \neq i} \frac{\gamma_{ij}}{|\vec{x}_i - \vec{x}_j|} = c_i$$

These are the same equations which determine the relative positions of the centers in stationary BPS solutions of $\mathcal{N} = 2$ supergravity, provided $c_i = 2\text{Im}(e^{-i\phi} Z(\gamma_i))$, $\phi = \arg(Z(\gamma))$.

- The moduli space

$$\mathcal{M}_n(\{\gamma_i, c_i\}) = \{(\vec{x}_i) \in \mathbb{R}^{3n}, \forall i \sum_{j \neq i} \frac{\gamma_{ij}}{|\vec{x}_i - \vec{x}_j|} = c_i\} / \mathbb{R}^3$$

carries a natural $SO(3)$ -invariant symplectic structure,

$$\omega = \frac{1}{4} \sum_{i < j} \epsilon_{abc} \frac{\gamma_{ij}}{r_{ij}^3} x_{ij}^a dx_{ij}^b \wedge dx_{ij}^c, \quad \vec{J} = \frac{1}{2} \sum_{i < j} \gamma_{ij} \frac{\vec{x}_{ij}}{r_{ij}},$$

- The Witten index localizes on time-independent configurations

[Girardello Imimbo Mukhi'83]

$$\mathcal{I}_n = \int \prod_{i=1}^{n-1} \frac{d^3 \vec{x}_i d\bar{\lambda}_i d\lambda_i dD_i}{4\pi^2 \beta} e^{-\beta(\sum_{i=1}^n (iU_i D_i + \sum_{i,j} (\frac{1}{2} M_{ij} D_i D_j + \vec{\nabla}_j U_i \bar{\lambda}_i \vec{\sigma} \lambda_j))}$$

where $M_{ij} = m_i \delta_{ij} - \frac{m_i m_j}{m_{\text{tot}}}$ is the reduced mass matrix.

- Integrating out the fermions produces

$$\int \prod_{i=1}^{n-1} d\bar{\lambda}_i d\lambda_i e^{-\beta \sum_{i,j=1}^{n-1} \vec{\nabla}_j U_i \bar{\lambda}_i \vec{\sigma} \lambda_j} = (\beta^2)^{n-1} \det(\vec{\nabla}_j U_i \otimes \vec{\sigma}),$$

Witten index from localization II

- Key observation: The bosonic configuration space $\mathbb{R}^{3n-3}$ is foliated by the phase spaces $\mathcal{M}_n(\{\gamma_i, u_i\})$ with $u_i \in \mathbb{R}^{n-1}$. The flat integration measure on $\mathbb{R}^{3n-3}$ combines with the fermionic determinant to produce the Liouville measure on $\mathcal{M}_n$, times flat measure on $\mathbb{R}^{n-1}$,

$$\prod_{i=1}^{n-1} d^3 \vec{x}_i \det(\vec{\nabla}_i U_j \otimes \vec{\sigma}) = \frac{(-1)^{n-1}}{2^{n-1} (n-1)!} \left(\prod_{i=1}^{n-1} du_i \right) \omega^{n-1},$$

Proof: follows from $\det(Q + iM) = \text{pf} \begin{pmatrix} M & Q \\ -Q & M \end{pmatrix}$.

- For $n = 2$, this boils down to $r^2 dr d\Omega_2 \times -\frac{\kappa^2}{4r^4} = \frac{1}{2} \kappa d\rho \times \frac{1}{2} \kappa d\Omega_2$.

Witten index from localization III

- Integrating over D_i , we get

$$\mathcal{I}_n = \sqrt{\det \frac{\beta}{8\pi M}} \int \prod_{i=1}^{n-1} du_i \operatorname{Vol}(\{\gamma_i, u_i\}) e^{-\frac{\beta}{8}(u_i - c_i) M_{ij}^{-1} (u_j - c_j)}$$

where $\operatorname{Vol}(\{\gamma_i, u_i\}) = \frac{(-1)^{\sum_{i < j} \gamma_{ij} - n + 1}}{(2\pi)^{n-1} (n-1)!} \int_{\mathcal{M}_n(\{\gamma_i, u_i\})} \omega^{n-1}$.

- The refined index is expected to be given by a similar formula, replacing the **symplectic volume** with the **equivariant Dirac index**,

$$\mathcal{I}_n = \sqrt{\det \frac{\beta}{8\pi M}} \int \prod_{i=1}^{n-1} du_i \operatorname{Ind}(\{\gamma_i, u_i\}, y) e^{-\frac{\beta}{8}(u_i - c_i) M_{ij}^{-1} (u_j - c_j)}$$

- At zero temperature, this is dominated by $u_i = c_i$, hence reduces to $\operatorname{Ind}(\{\gamma_i, c_i\}, y)$ counting supersymmetric bound states.

- Both $\text{Vol}(\{\gamma_i, u_i\})$ and $\text{Ind}(\{\gamma_i, u_i\}, y)$ are locally constant functions of u_i , away from walls of marginal stability. Thus the Witten index is a linear combination of generalized error functions !
- Moreover, both are computable using Duistermaat-Heckman / Atiyah-Bott localization with respect to $U(1) \subset SO(3)$. [*Manschot BP Sen'11*]:

$$\text{Ind}(\{\gamma, \mathbf{c}\}; y) = \frac{(-1)^{\sum_{i < j} \gamma_{ij} - n + 1}}{(y - 1/y)^{n-1}} \sum_{\sigma \in S_n} F_n(\{\gamma_{\sigma(i)}, \mathbf{c}_{\sigma(i)}\}) y^{\sum_{i < j} \gamma_{\sigma(i)\sigma(j)}}$$

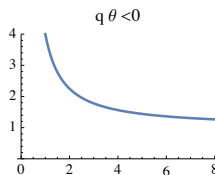
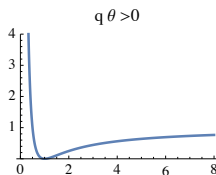
$$\text{Vol}(\{\gamma_i, \mathbf{c}_i\}) = \lim_{y \rightarrow 1} \text{Ind}(\{\gamma, \mathbf{c}\}; y)$$

Two-body electron-monopole problem I

- Consider a non-relativistic particle of electric charge $q = \frac{1}{2} \langle \gamma_1, \gamma_2 \rangle$ in the field of a Dirac monopole of unit magnetic charge:

$$H = \frac{1}{2m} (\vec{p} - q\vec{A})^2 - \frac{q}{2m} \vec{B} \cdot \vec{\sigma} \otimes (1_2 - \sigma_3) + \frac{1}{2m} \left(\vartheta - \frac{q}{r} \right)^2$$

$$\vec{\nabla} \wedge \vec{A} = \vec{B} = \frac{\vec{r}}{r^3}, \quad m = \frac{|Z_{\gamma_1}| |Z_{\gamma_2}|}{|Z_{\gamma_1}| + |Z_{\gamma_2}|}, \quad \frac{\vartheta^2}{2m} = |Z_{\gamma_1}| + |Z_{\gamma_2}| - |Z_{\gamma_1 + \gamma_2}|$$



Two-body electron-monopole problem II

- H describes two bosonic degrees of freedom with helicity $h = 0$, and one helicity $h = \pm 1/2$ fermionic doublet with gyromagnetic ratio $g = 4$.

D'Hoker Vinet 1985; Denef 2002; Avery Michelson 2007; Lee Yi 2011

- H commutes with 4 supercharges – here $\vec{\Pi} = \vec{p} - q\vec{A}$:

$$Q_4 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 & -i(\vartheta - \frac{q}{r}) + \vec{\sigma} \cdot \vec{\Pi} \\ i(\vartheta - \frac{q}{r}) + \vec{\sigma} \cdot \vec{\Pi} & 0 \end{pmatrix}$$

$$Q_a = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 & -(\vartheta - \frac{q}{r}) \vec{\sigma} - i\vec{\Pi} + \vec{\Pi} \wedge \vec{\sigma} \\ -(\vartheta - \frac{q}{r}) \vec{\sigma} + i\vec{\Pi} + \vec{\Pi} \wedge \vec{\sigma} & 0 \end{pmatrix}.$$

$$\{Q_m, Q_n\} = 2H \delta_{mn}$$

Two-body electron-monopole problem III

- Going to a basis of monopole spherical harmonics, the Schrödinger equation with energy $E = k^2/(2m)$ becomes

$$\left[-\frac{1}{r} \partial_r^2 r + \frac{\nu^2 - q^2 - \frac{1}{4}}{r^2} + \left(\vartheta - \frac{q}{r} \right)^2 \right] \Psi(r) = k^2 \Psi,$$

where

$$\nu = j + \frac{1}{2} + h, \quad j = |q| + h + \ell, \ell \in \mathbb{N}.$$

- Supersymmetric bound states exist for $q\vartheta > 0$, $h = -1/2$, $\ell = 0$, and form a multiplet of spin $j = |q| - \frac{1}{2}$, with $2j + 1 = |\langle \gamma_1, \gamma_2 \rangle|$.

Denef 2002

Two-body electron-monopole problem IV

- The S-matrix for partial waves is similar to that of H-atom,

$$S_\nu(k) = \frac{\Gamma\left(\frac{1}{2} + \nu + i\frac{q\vartheta}{\sqrt{k^2 - \vartheta^2}}\right)}{\Gamma\left(\frac{1}{2} + \nu - i\frac{q\vartheta}{\sqrt{k^2 - \vartheta^2}}\right)} = e^{2i\delta_\nu(k)}.$$

BP, arXiv:1501.01643

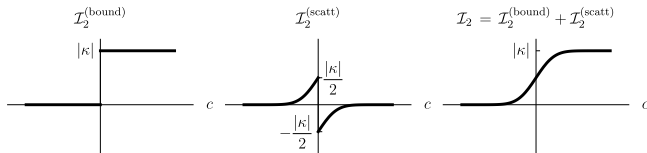
- The contribution of the continuum to $\text{Tr}(-1)^F e^{-2\pi RH}$ is thus

$$\sum_{h=0^2, \pm\frac{1}{2}} (-1)^{2h} \sum_{\ell=0}^{\infty} \int_{k=\vartheta}^{\infty} \frac{dk}{2\pi i} \partial_k \log \frac{\Gamma\left(|q| + \ell + 2h + 1 + i\frac{q\vartheta}{\sqrt{k^2 - \vartheta^2}}\right)}{\Gamma\left(|q| + \ell + 2h + 1 - i\frac{q\vartheta}{\sqrt{k^2 - \vartheta^2}}\right)} e^{-\frac{\pi Rk^2}{m}}$$

Two-body electron-monopole problem V

- Terms with $\ell > 0$ cancel, leaving the contribution from $\ell = 0$ only:

$$\begin{aligned}
 \text{Tr}(-1)^F e^{-2\pi RH} &= -|2q| \Theta(q\vartheta) - \frac{2q\vartheta}{\pi} \int_{k=|\vartheta|}^{\infty} \frac{dk}{k\sqrt{k^2 - \vartheta^2}} e^{-\frac{\pi Rk^2}{m}} \\
 &= -2|q| \Theta(q\vartheta) + |q| \text{sgn}(q\vartheta) \text{Erfc} \left(|\vartheta| \sqrt{\frac{\pi R}{m}} \right) \\
 &= -|q| - q \text{Erf} \left(\vartheta \sqrt{\frac{\pi R}{m}} \right) .
 \end{aligned}$$



Two-body electron-monopole problem VI

- Using localization, this is easily reproduced:

$$\begin{aligned}\mathcal{I}_2 &= -\frac{\beta\kappa^2}{4\pi} \int_0^\infty \frac{r^2 dr}{r^4} \int_{-\infty}^\infty dD \, e^{-\beta\left(-\frac{i}{2}D\left(\frac{\kappa}{r}-c\right)+\frac{m}{2}D^2\right)} \\ &= -\kappa^2 \left(\frac{\beta}{8\pi m}\right)^{\frac{1}{2}} \int_0^\infty d\rho \, e^{-\frac{\beta}{8m}(\kappa\rho-c)^2} \\ &= -\frac{\kappa^2}{2} \left(\frac{\beta}{8\pi m}\right)^{\frac{1}{2}} \int_{-\infty}^\infty d\rho (1 + \text{sign } \rho) \, e^{-\frac{\beta}{8m}(\kappa\rho-c)^2} \\ &= -\frac{\kappa}{2} \left[\text{sign}(\kappa) + E_1 \left(c \sqrt{\frac{\beta}{8\pi m}} \right) \right]\end{aligned}$$

with $\rho = 1/r$.

- When $\mathcal{M}_n(\{\gamma_i, u_i\})$ is **compact**, its equivariant volume or Dirac index is computable by localization with respect to $U(1) \subset SO(3)$. Fixed points are collinear configurations subject to

$$\sum_{j \neq i} \frac{\gamma_{ij}}{|z_i - z_j|} = c_i,$$

i.e. critical points of $W = -\sum_{i < j} \gamma_{ij} \log |z_j - z_i| - \sum_i c_i z_i$.

- Solutions are classified by the ordering of the centers along the z -axis, weighted by $\text{sign det } \partial^2 W$:

$$\text{Ind}_C(\{\gamma, c\}; y) = \frac{(-1)^{\sum_{i < j} \gamma_{ij} - n + 1}}{(y-1/y)^{n-1}} \sum_{\sigma \in S_n} F_{C,n}(\{\gamma_{\sigma(i)}, c_{\sigma(i)}\}) y^{\sum_{i < j} \gamma_{\sigma(i)\sigma(j)}}$$

BPS index from localization II

- When $\mathcal{M}_n(\{\gamma_i, u_i\})$ is not compact, i.e. in the presence of scaling solutions, there are additional boundary contributions which ensure that $\text{Ind}(\{\gamma, c\}; y)$ is a symmetric Laurent polynomial in y .
- Alternatively, use the flow tree formula.

$$\text{Ind}_{\text{tree}}(\{\gamma, c\}; y) = \frac{(-1)^{\sum_{i < j} \gamma_{ij} - n + 1}}{(y - 1/y)^{n-1}} \sum_{\sigma \in S_n} F_{\text{tree}, n}(\{\gamma_{\sigma(i)}, c_{\sigma(i)}\}) y^{\sum_{i < j} \gamma_{\sigma(i)\sigma(j)}}$$

where $F_{\text{tree}, n}(\{\gamma_i, c_i\})$ counts planar attractor flow trees. E.g.

$$F_{\text{tree}, 2} = \frac{1}{2} [\text{sgn}(c_1) + \text{sgn}(\gamma_{12})]$$

$$F_{\text{tree}, 3} = \frac{1}{4} [(\text{sgn}(c_1) + \text{sgn}(\gamma_{12})) (\text{sgn}(c_1 + c_2) + \text{sgn}(\gamma_{23})) \\ - (\text{sgn}(\gamma_{2+3,1}) + \text{sgn}(\gamma_{12})) (\text{sgn}(\gamma_{3,1+2}) + \text{sgn}(\gamma_{23}))],$$

- Using this, we find e.g. the partial Witten index for 3 centers

$$\begin{aligned} \mathcal{I}_3 = & \frac{1}{4} \left[E_2 \left(\sqrt{\frac{m_1 m_3}{m_2(m_1+m_2+m_3)}}; \frac{c_2 m_1 - c_1 m_2}{\sqrt{m_1 m_2(m_1+m_2)}}, \frac{c_3}{\sqrt{m_{1+2,3}}} \right) \right. \\ & - \operatorname{sgn}(\gamma_{1,2+3}) \operatorname{sgn}(\gamma_{1+2,3}) \\ & - \left[E_1 \left(\frac{c_3}{\sqrt{m_{1+2,3}}} \right) - \operatorname{sgn}(\gamma_{1+2,3}) \right] \operatorname{sgn}(\gamma_{12}) \\ & \left. + \left[E_1 \left(\frac{c_1}{\sqrt{m_{1,2+3}}} \right) - \operatorname{sgn}(\gamma_{2+3,1}) \right] \operatorname{sgn}(\gamma_{23}) \right]. \end{aligned}$$

where $m_{1+2,3} = \frac{m_3(m_1+m_2)}{(m_1+m_2+m_3)}$, $\gamma_{1+2,3} = \gamma_1 + \gamma_2$.

- $\mathcal{I}_3(\beta, y)$ is obtained by rescaling $m \mapsto 8\pi m/\beta$, multiplying by $y^{\gamma_{12}+\gamma_{23}+\gamma_{13}}/(y-1/y)^2$ and summing over permutations.

- Setting $\beta = 2\pi\tau_2$, this matches the error functions appearing in the 'instanton generating potential', which plays the role of Witten index in 4D [*Alexandrov Moore Neizke BP'14*]

$$\mathcal{G} = \sum_{n=1}^{\infty} \frac{1}{2\pi\sqrt{\tau_2}} e^{-\sum_i S_{p_i}^{\text{cl}}} \vartheta_{p,\mu}(\Phi_n^{\text{tot}}, -1) \left[\prod_{i=1}^n \sum_{p_i, \mu_i} h_{p_i, \mu_i} \right]$$

- For one-parameter models, or more generally for collinear magnetic charges, the same error functions evaluated at the large volume attractor point appear in the modular completion of $h_{p,\mu}$. In general however, the x -arguments differ by a rescaling, e.g a factor $\sqrt{\frac{(p^3)(p_1 p_2 p)}{(p_1 p^2)(p_2 p^2)}}$ in the 2 body case.

- At the large volume attractor point, only contributions from the continuum of scattering states remain:

$$\mathcal{J}_3^* = \frac{1}{4} \left(M_2 \left(\sqrt{\frac{m_1 m_3}{m_2(m_1+m_2+m_3)}}; -\frac{\sqrt{\tau_2}(m_2\gamma_{1,2+3}-m_1\gamma_{2,1+3})}{\sqrt{m_1 m_2(m_1+m_2)}}, -\frac{\sqrt{\tau_2}\gamma_{1+2,3}}{\sqrt{m_{1+2,3}}} \right) \right. \\ \left. + M_1 \left(\frac{\sqrt{\tau_2}\gamma_{1+2,3}}{\sqrt{m_{1+2,3}}} \right) (\text{sgn}(m_2\gamma_{1,2+3}-m_1\gamma_{2,1+3}) - \text{sgn}(\gamma_{12})) \right. \\ \left. + M_1 \left(\frac{\sqrt{\tau_2}\gamma_{1,2+3}}{\sqrt{m_{1,2+3}}} \right) (\text{sgn}(m_2\gamma_{1+2,3}-m_3\gamma_{1+3,2}) - \text{sgn}(\gamma_{23})) \right).$$

with $\beta = 2\pi\tau_2$.

- For collinear magnetic charges (e.g. in one-parameter models), the coefficients of M_1 vanish, leaving only the contribution from genuine 3-body scattering.

Summary and open problems I

- Using localization, we managed to evaluate the Witten index of the quantum mechanics of n dyons, including both bound state and continuum contributions. Compare with the unsolvable n -body problem in Newtonian gravity !
- Our derivation reproduces the non-holomorphic terms in the modular completion of the generating series of D4-D2-D0 invariants, that were predicted earlier using indirect arguments.
- Some details still need to be clarified, e.g. the η -deformation needed to resolve the ambiguity in $\text{sign}(0)$, Kronecker delta contributions supported on walls, a proper derivation of the refined index, etc.
- Currently the modular completion has mainly been tested in one-modulus examples. It would be interesting to study examples with two moduli, e.g. genus-one fibrations or K3 fibrations.

Thanks for your attention !

